Advances in Mathematics 331 (2018) 627–719

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On Nash images of Euclidean spaces

José F. Fernando [1]

Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas, Universidad Complutense de Madrid, 28040 Madrid, Spain

A R T I C L E   I N F O

Article history:
Received 7 February 2017
Received in revised form 12 February 2018
Accepted 2 April 2018
Communicated by A. Asok

Dedicated to Prof. Masahiro Shiota, in memoriam

MSC:
primary 14P20, 58A07
secondary 14P10, 14E15

Keywords:
Nash image
Arc-symmetric semialgebraic set
Desingularization
Drilling blow-up
Well-welded semialgebraic set
Nash path-connected semialgebraic set

A B S T R A C T

In this work we characterize the subsets of $\mathbb{R}^n$ that are images of Nash maps $f : \mathbb{R}^m \to \mathbb{R}^n$. We prove Shiota's conjecture and show that *a subset $\mathcal{S} \subset \mathbb{R}^n$ is the image of a Nash map $f : \mathbb{R}^m \to \mathbb{R}^n$ if and only if $\mathcal{S}$ is semialgebraic, pure dimensional of dimension $d \leq m$ and there exists an analytic path $\alpha : [0, 1] \to \mathcal{S}$ whose image meets all the connected components of the set of regular points of $\mathcal{S}$.* Two remarkable consequences are the following: (1) pure dimensional irreducible semialgebraic sets of dimension d with arc-symmetric closure are Nash images of $\mathbb{R}^d$; and (2) semialgebraic sets are projections of irreducible algebraic sets whose connected components are Nash diffeomorphic to Euclidean spaces.

E-mail address: josefer@mat.ucm.es.

[1] Author supported by Spanish MINECO grant GRAYAS MTM2014-55565-P, Spanish MINECO grant STRANO MTM2017-82105-P and Grupos UCM 910444. This article has been finished during a one year research stay of the author in the Dipartimento di Matematica of the Università di Pisa. The author would like to thank the department for the invitation and the very pleasant working conditions. The 1-year research stay of the author was partially supported by Spanish MECD grant PRX14/00016.

https://doi.org/10.1016/j.aim.2018.04.011

Contents

1. Introduction

Although it is usually said that the first work in Real Geometry is due to Harnack [23], who obtained an upper bound for the number of connected components of a non-singular real algebraic curve in terms of its genus, modern Real Algebraic Geometry was born with Tarski's article [36], where it is proved that the image of a semialgebraic set under a polynomial map is a semialgebraic set. A map $f := (f_1, \dots, f_n) : \mathbb{R}^m \to \mathbb{R}^n$ is *polynomial* if its components $f_k \in \mathbb{R}[\mathtt{x}] := \mathbb{R}[\mathtt{x}_1, \dots, \mathtt{x}_m]$ are polynomials. Analogously, f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials $g_k, h_k \in \mathbb{R}[\mathtt{x}]$ such that h_k never vanishes on $\mathbb{R}^m$. A subset $\mathcal{S} \subset \mathbb{R}^n$ is *semialgebraic* when it has a description by a finite boolean combination of polynomial equalities and inequalities, which we will call a *semialgebraic* description. Unless stated otherwise, the topology employed in the article is the Euclidean one.

We are interested in studying what might be called the 'inverse problem' to Tarski's result. In the 1990 *Oberwolfach reelle algebraische Geometrie* week [22] Gamboa proposed:

Problem 1.1. *To characterize the (semialgebraic) subsets of $\mathbb{R}^n$ that are either polynomial or regular images of $\mathbb{R}^m$.*

During the last decade we have attempted to understand better polynomial and regular images of $\mathbb{R}^m$. Our main objectives have been the following:

- To find obstructions to be either polynomial or regular images.
- To prove (constructively) that large families of semialgebraic sets with piecewise linear boundary (convex polyhedra, their interiors, complements and the interiors of their complements) are either polynomial or regular images of Euclidean spaces.