

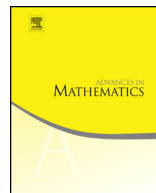


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On Nash images of Euclidean spaces

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ABSTRACT

In this work we characterize the subsets of $\mathbb{R}^n$ that are images of Nash maps $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$. We prove Shiota's conjecture and show that a subset $S \subset \mathbb{R}^n$ is the image of a Nash map $f : \mathbb{R}^m \rightarrow \mathbb{R}^n$ if and only if S is semialgebraic, pure dimensional of dimension $d \leq m$ and there exists an analytic path $\alpha : [0, 1] \rightarrow S$ whose image meets all the connected components of the set of regular points of S . Two remarkable consequences are the following: (1) pure dimensional irreducible semialgebraic sets of dimension d with arc-symmetric closure are Nash images of $\mathbb{R}^d$; and (2) semialgebraic sets are projections of irreducible algebraic sets whose connected components are Nash diffeomorphic to Euclidean spaces.

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1. Introduction

Although it is usually said that the first work in Real Geometry is due to Harnack [23], who obtained an upper bound for the number of connected components of a non-singular real algebraic curve in terms of its genus, modern Real Algebraic Geometry was born with Tarski's article [36], where it is proved that the image of a semialgebraic set under a polynomial map is a semialgebraic set. A map $f := (f_1, \dots, f_n) : \mathbb{R}^m \rightarrow \mathbb{R}^n$ is *polynomial* if its components $f_k \in \mathbb{R}[\mathbf{x}] := \mathbb{R}[x_1, \dots, x_m]$ are polynomials. Analogously, f is *regular* if its components can be represented as quotients $f_k = \frac{g_k}{h_k}$ of two polynomials $g_k, h_k \in \mathbb{R}[\mathbf{x}]$ such that h_k never vanishes on $\mathbb{R}^m$. A subset $S \subset \mathbb{R}^n$ is *semialgebraic* when it has a description by a finite boolean combination of polynomial equalities and inequalities, which we will call a *semialgebraic* description. Unless stated otherwise, the topology employed in the article is the Euclidean one.

We are interested in studying what might be called the ‘inverse problem’ to Tarski's result. In the 1990 *Oberwolfach reelle algebraische Geometrie* week [22] Gamboa proposed:

Problem 1.1. *To characterize the (semialgebraic) subsets of $\mathbb{R}^n$ that are either polynomial or regular images of $\mathbb{R}^m$.*

During the last decade we have attempted to understand better polynomial and regular images of $\mathbb{R}^m$. Our main objectives have been the following:

- To find obstructions to be either polynomial or regular images.
- To prove (constructively) that large families of semialgebraic sets with piecewise linear boundary (convex polyhedra, their interiors, complements and the interiors of their complements) are either polynomial or regular images of Euclidean spaces.

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