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A Crossing Lemma for Jordan curves [☆]

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ABSTRACT

If two Jordan curves in the plane have precisely one point in common, and there they do not properly cross, then the common point is called a *touching point*. The main result of this paper is a Crossing Lemma for simple curves: Let X and T stand for the sets of intersection points and touching points, respectively, in a family of n simple curves in the plane, no three of which pass through the same point. If $|T| > cn$, for some fixed constant $c > 0$, then we prove that $|X| = \Omega(|T|(\log\log(|T|/n))^{1/504})$. In particular, if $|T|/n \to \infty$, then the number of intersection points is much larger than the number of touching points.
As a corollary, we confirm the following long-standing conjecture of Richter and Thomassen: The total number of intersection points between n pairwise intersecting simple closed (i.e.,

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Jordan) curves in the plane, no three of which pass through the same point, is at least $(1 - o(1))n^2$.

1. Introduction

1.1. Preliminaries

Arrangements of curves and surfaces. It was a fruitful and surprising discovery made in the 1980s that the Piano Mover's Problem and many other algorithmic and optimization questions in motion planning, ray shooting, computer graphics etc., boil down to computing certain elementary substructures (e.g., cells, envelopes, k-levels, or zones) in arrangements of curves in the plane and surfaces in higher dimensions [14,29,39,43]. Hence, the performance of the most efficient algorithms for the solution of such problems is typically determined by the *combinatorial complexity* of a single cell or a collection of several cells in the underlying arrangement, that is, the total number of their faces of all dimensions.

The study of arrangements has brought about a renaissance of Erdős-type combinatorial geometry. For instance, in the plane, Erdős's famous question [17] on the maximum number of times the unit distance can occur among n points in the plane can be generalized as follows [10]: *What is the maximum total number of sides of n cells in an arrangement of n unit circles in the plane?* In the limiting case, when k circles pass through the same point p (which is, therefore, at unit distance from k circle centers), p can be regarded as a degenerate cell with k sides.

Several beautiful paradigms have emerged as a result of this interplay between combinatorial and computational geometry, from the random sampling argument of Clarkson and Shor [11] through epsilon-nets (Haussler–Welzl [28]) to the discrepancy method (Chazelle [9]). It is worth noting that most of these tools are restricted to families of curves and surfaces of *bounded description complexity*. This roughly means that a curve in the family can be given by a bounded number of reals (like the coefficients of a bounded degree polynomial). For the exact definition, see [43].

Another tool that proved to be applicable to Erdős's questions on repeated distances is the *Crossing Lemma* of Ajtai, Chvátal, Newborn, Szemerédi and Leighton [3,32]. It states that no matter how we a draw a sufficiently dense graph $G = (V, E)$ in the plane or on a fixed surface, the number of crossings between its edges is at least $\Omega(|E|^3/|V|^2)$.

In particular, this implies that if G has a lot more edges than vertices (that is, $|E|/n \to \infty$), then its number of crossings is much larger than its number of edges. The best known upper bound on the k-set problem [12], needed for the analysis of many important geometric algorithms, and the most elegant proofs of the Szemerédi–Trotter theorem [45], [46] on the maximum number of incidences between a set of points and a set of lines (or