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# On geometric quantization of $b$-symplectic manifolds

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A R T I C L E  I N F O

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A B S T R A C T

We study a notion of pre-quantization for $b$-symplectic manifolds. We use it to construct a formal geometric quantization of $b$-symplectic manifolds equipped with Hamiltonian torus actions with nonzero modular weight. We show that these quantizations are finite dimensional $T$-modules.



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## 1. Introduction

Let $(M, \omega)$ be an integral symplectic manifold, and let $(\mathcal{L}, \nabla)$ be a line bundle $\mathcal{L}$ with connection $\nabla$ of curvature $\omega$. The quadruple $(M, \omega, \mathcal{L}, \nabla)$ is called a *prequantization* [6] of $(M, \omega)$, which morally should give rise to a geometric quantization $Q(M)$ of $M$. A complication arises in that all known constructions of $Q(M)$ require additional data, a polarization of $M$; such a polarization may be real, a foliation of $M$ by Lagrangian subvarieties, or else complex, a complex or almost complex structure on $M$ compatible with $\omega$. It is generally believed, and in many cases verified, that the quantization $Q(M)$ should be independent of the polarization. However there is no theorem guaranteeing that this should be the case.

Work of Kontsevich [5] extending deformation quantization to Poisson manifolds raises the issue as to whether any of the constructions above has any relevance in the Poisson setting. If $(M, \pi)$ is a Poisson manifold, it is not clear what the analog of $(\mathcal{L}, \nabla)$ should be, let alone what one would mean by a polarization.[3] The purpose of this paper is to try to begin developing some examples, guided by symplectic geometry, where a sensible theory of geometric quantization of Poisson manifolds can be proposed. Hopefully the repertoire of examples may be a guide to a theory of geometric quantization of Poisson manifolds.

To do this we focus on a special class of Poisson manifolds that have two helpful properties. First, we require that the Poisson structure be symplectic on the complement of a real hypersurface $Z \subset M$ and be minimally degenerate (in the sense that the top alternating power of the Poisson bivector field has a simple zero) on $Z$. Such $b$-symplectic manifolds have been the subject of intensive study [2,4] and are by now well understood geometrically. And second, we require that the manifold have a Hamiltonian action of a torus with a certain nondegeneracy condition (nonzero modular weight; see Theorem 3.5 below for the precise definition). The presence of these two conditions allows us to bring tools from symplectic geometry to bear on the problem. One concept which, as far as we know, has not been investigated at all in the $b$-symplectic setting and which will play an essential ingredient in describing how to quantize these manifolds, is the concept of "integrality" for the $b$-symplectic form $\omega$ (or, alternatively of "pre-quantizability" for the pair, $(M, \omega)$); and one of the main goals of this paper will be to provide an appropriate definition of this concept and explore some of its consequences. We then show that a natural functoriality condition for quantization ("formal geometric quantization") determines what the quantization of the manifold should be.

Formal geometric quantization was studied in [14,9] in the context of the quantization of Hamiltonian $T$-spaces with proper moment map. We will see that where $M$ is a compact $b$-symplectic manifold, with a Hamiltonian torus action of nonzero modular weight, the manifold $M - Z$ is such a space, and that an analog of formal geometric quantization for $b$-symplectic manifolds yields essentially the quantization of $M - Z$. However, in the

[3] See [11] for an alternative approach to the prequantization of Poisson manifolds.