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Decomposition spaces, incidence algebras and Möbius inversion I: Basic theory ☆

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ABSTRACT

This is the first in a series of papers devoted to the theory of decomposition spaces, a general framework for incidence algebras and Möbius inversion, where algebraic identities are realised by taking homotopy cardinality of equivalences of ∞-groupoids. A decomposition space is a simplicial ∞-groupoid satisfying an exactness condition, weaker than the Segal condition, expressed in terms of active and inert maps in $\mathbb{\Delta}$. Just as the Segal condition expresses composition, the new exactness condition expresses decomposition, and there is an abundance of examples in combinatorics.
After establishing some basic properties of decomposition spaces, the main result of this first paper shows that to any decomposition space there is an associated incidence coalgebra, spanned by the space of 1-simplices, and with coefficients in ∞-groupoids. We take a functorial viewpoint throughout, emphasising conservative ULF functors; these induce coalgebra homomorphisms. Reduction procedures in the classical theory of incidence coalgebras are examples of this notion, and many are examples of decalage of decomposition spaces. An interesting class of examples of decomposition spaces beyond Segal spaces is provided by Hall algebras: the Waldhausen

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$S_\bullet$-construction of an abelian (or stable infinity) category is shown to be a decomposition space.
In the second paper in this series we impose further conditions on decomposition spaces, to obtain a general Möbius inversion principle, and to ensure that the various constructions and results admit a homotopy cardinality. In the third paper we show that the Lawvere–Menni Hopf algebra of Möbius intervals is the homotopy cardinality of a certain universal decomposition space. Two further sequel papers deal with numerous examples from combinatorics.
Note: The notion of decomposition space was arrived at independently by Dyckerhoff and Kapranov [17] who call them unital 2-Segal spaces. Our theory is quite orthogonal to theirs: the definitions are different in spirit and appearance, and the theories differ in terms of motivation, examples, and directions.

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Note. This paper originally formed Sections 1 and 2 of the manuscript [21], which has now been split into six papers.

0. Introduction

The notion of incidence algebra of a locally finite poset is an important construction in algebraic combinatorics, with applications to many fields of mathematics. In this work we generalise this construction in three directions: (1) we replace posets by categories and ∞-categories; (2) we replace scalar coefficients in a field by ∞-groupoids, working at the objective level, ensuring natively bijective proofs [22]; and most importantly: (3) we replace the Segal condition, which essentially characterises ∞-categories among simplicial ∞-groupoids, by a weaker condition that still allows the construction of incidence algebras. Simplicial ∞-groupoids satisfying this axiom are called decomposition