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Isotropic constants and Mahler volumes

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ABSTRACT

This paper contains a number of results related to volumes of projective perturbations of convex bodies and the Laplace transform on convex cones. First, it is shown that a sharp version of Bourgain's slicing conjecture implies the Mahler conjecture for convex bodies that are not necessarily centrally-symmetric. Second, we find that by slightly translating the polar of a centered convex body, we may obtain another body with a bounded isotropic constant. Third, we provide a counter-example to a conjecture by Kuperberg on the distribution of volume in a body and in its polar.

1. Introduction

This paper describes interrelations between duality and distribution of volume in convex bodies. A convex body is a compact, convex subset $K \subseteq \mathbb{R}^n$ whose interior Int(K) is non-empty. If $0 \in \mathrm{Int}(K)$, then the polar body is defined by

$$K^\circ = \{y \in \mathbb{R}^n\, ;\, \forall x \in K,\ \langle x, y \rangle \leq 1\}.$$

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The polar body K° is itself a convex body with the origin in its interior, and moreover $(K^\circ)^\circ = K$. The *Mahler volume* of a convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior is defined as

$$s(K) = Vol_n(K) \cdot Vol_n(K^\circ),$$

where Vol_n is n-dimensional volume. In the class of convex bodies with barycenter at the origin, the Mahler volume is maximized for ellipsoids, as proven by Santaló [29], see also Meyer and Pajor [20]. The Mahler conjecture suggests that for any convex body $K \subseteq \mathbb{R}^n$ containing the origin in its interior,

$$s(K) \geq s(\Delta^n) = \frac{(n+1)^{n+1}}{(n!)^2}, \tag{1}$$

where $\Delta^n \subseteq \mathbb{R}^n$ is any simplex whose vertices span $\mathbb{R}^n$ and add up to zero. The conjecture was verified for convex bodies with certain symmetries in the works of Barthe and Fradelizi [3], Kuperberg [14] and Saint Raymond [28]. In two dimensions the conjecture was proven by Mahler [18], see also Meyer [19]. There is also a well-known version of the Mahler conjecture for centrally-symmetric convex bodies (i.e., when $K = -K$) that will not be discussed much here. It was proven by Bourgain and Milman [8] that for any convex body $K \subseteq \mathbb{R}^n$ with the origin in its interior,

$$s(K) \geq c^n \cdot s(\Delta^n) \tag{2}$$

for some universal constant $c > 0$. There are several beautiful, completely different proofs of the Bourgain–Milman inequality in addition to the original argument, including proofs by Kuperberg [14], by Nazarov [23] and by Giannopoulos, Paouris and Vritsiou [11]. The covariance matrix of a convex body $K \subseteq \mathbb{R}^n$ is the matrix $\mathrm{Cov(K)} = (\mathrm{Cov_{ij}})_{\mathrm{i,j=1,\dots,n}}$ where

$$\mathrm{Cov_{ij}} = \frac{\int_K x_i x_j dx}{\mathrm{Vol_n(K)}} - \frac{\int_K x_i}{\mathrm{Vol_n(K)}} \cdot \frac{\int_K x_j}{\mathrm{Vol_n(K)}}.$$

The isotropic constant of a convex body $K \subseteq \mathbb{R}^n$ is the parameter $L_K > 0$ defined via

$$L_K^{2n} = \frac{\det \mathrm{Cov(K)}}{Vol_n(K)^2}. \tag{3}$$

We equip the space of convex bodies in $\mathbb{R}^n$ with the usual Hausdorff topology. The Mahler volume is a continuous function defined on the class of convex bodies in $\mathbb{R}^n$ containing the origin in their interior. A standard compactness argument shows that the minimum of the Mahler volume in this class is indeed attained. Below we present a variational argument in the class of projective images of K that yields the following: