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Asymptotic joint distribution of the extremities of a random Young diagram and enumeration of graphical partitions

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A R T I C L E   I N F O

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A B S T R A C T

An integer partition of n is a decreasing sequence of positive integers that add up to $[n]$. Back in 1979 Macdonald posed a question about the limit value of the probability that two partitions chosen uniformly at random, and independently of each other, are comparable in terms of the dominance order. In 1982 Wilf conjectured that the uniformly random partition is a size-ordered degree sequence of a simple graph with the limit probability 0. In 1997 we showed that in both, seemingly unrelated, cases the limit probabilities are indeed zero, but our method left open the problem of convergence rates. The main result in this paper is that each of the probabilities is $e^{-0.11 \log n/\log\log n}$, at most. A key element of the argument is a local limit theorem, with convergence rate, for the *joint* distribution of the $[n^{1/4-\varepsilon}]$ tallest columns and the $[n^{1/4-\varepsilon}]$ longest rows of the Young diagram representing the random partition.

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1. Introduction and main results

A weakly decreasing sequence $\lambda = (\lambda_1, \dots, \lambda_m)$, $m = m(\lambda) \ge 1$, of positive integers is called a partition of a positive integer n into m parts if $\lambda_1 + \dots + \lambda_m = n$. We will denote the set of all such partitions λ by Ω_n. It is customary to visualize a partition λ as a (Young–Ferrers) diagram formed by n unit squares, with the columns of decreasing heights $\lambda_1, \dots, \lambda_m$. We will use the same letter λ for the diagram representing the partition λ. Introduce the positive integers

$$\lambda'_i = \big|\{1 \le j \le m(\lambda) : \lambda_j \ge i\}\big|, \quad 1 \le i \le \lambda_1;$$

so λ'_i is the number of parts in the partition λ that are i, at least. Clearly λ'_i decrease and add up to n; so $\lambda' := (\lambda'_1, \dots, \lambda'_{m'})$, $(m' = \lambda_1)$, is a partition of $[n]$, usually referred to as the conjugate to λ.

The dominance order on the set Ω_n is a partial order $\preceq$ defined as follows. For $\lambda, \mu \in \Omega_n$, we write $\lambda \preceq \mu$ if

$$\sum_{j=1}^{i} \lambda_j \le \sum_{j=1}^{i} \mu_j, \quad i \ge 1; \tag{1}$$

(by definition $\lambda_j = 0$ for $i > m(\lambda)$, $\mu_j = 0$ for $j > m(\mu)$). Under $\preceq$, Ω_n is a lattice. Brylawski [5] demonstrated how ubiquitous this lattice is. For instance, Gale–Ryser theorem (Gale [11] and Ryser [21], Brualdi and Ryser [4]) asserts: given two decreasing positive tuples $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_r)$, $\beta = (\beta_1, \beta_2, \dots, \beta_s)$, there exists a *bipartite* graph on a vertex set (X, Y), $|X| = r$, $|Y| = s$, such α and β are the size-ordered degree sequences of vertices in X and Y respectively, iff $\sum_{t=1}^{r} \alpha_t = \sum_{t=1}^{s} \beta_t$ and $\alpha \preceq \beta'$. The lattice Ω_n is also at the core of the classic description of the irreducible representations of the symmetric group S_n, see Diaconis [6], Macdonald [14], Sagan [22], for instance.

Soon after [11], [21], Erdős and Gallai [7] found the necessary and sufficient conditions a partition $\lambda \in \Omega_n$ (n even) has to satisfy to be *graphical*, i.e. to be a size-ordered degree sequence of a simple graph. According to Nash–Williams (see Sierksma and Hoogeveen [23] for the proof), the Erdős–Gallai conditions are equivalent to

$$\sum_{j=1}^{i} \lambda'_j \ge \sum_{j=1}^{i} \lambda_j + i, \quad 1 \le i \le D(\lambda); \tag{2}$$

here $D(\lambda)$ is the size of the Durfee square of λ, i.e. the number of rows of the largest square inscribed into the Young diagram λ. Obvious differences notwithstanding, the Gale–Ryser conditions and the Nash–Williams conditions are undeniably similar.

Macdonald [14] (Ch. 1, Section 1, Example 18) posed a probabilistic question, which may be formally interpreted as follows. Let $\lambda, \mu \in \Omega_n$ be chosen uniformly at random