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Enriched Stone-type dualities ☆

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A B S T R A C T

A common feature of many duality results is that the involved equivalence functors are liftings of hom-functors into the two-element space resp. lattice. Due to this fact, we can only expect dualities for categories cogenerated by the two-element set with an appropriate structure. A prime example of such a situation is Stone's duality theorem for Boolean algebras and Boolean spaces, the latter being precisely those compact Hausdorff spaces which are cogenerated by the two-element discrete space. In this paper we aim for a systematic way of extending this duality theorem to categories including all compact Hausdorff spaces. To achieve this goal, we combine duality theory and quantale-enriched category theory. Our main idea is that, when passing from the two-element discrete space to a cogenerator of the category of compact Hausdorff spaces, all other involved structures should be substituted by corresponding enriched versions. Accordingly, we work with the unit interval $[0, 1]$ and present duality theory for ordered and metric compact Hausdorff spaces and (suitably defined) finitely cocomplete categories enriched in $[0, 1]$.

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Contents

1. Introduction

In [5], the authors make the seemingly paradoxical observation that "...an equation is only interesting or useful to the extent that the two sides are different!". Undoubtedly, a moment's thought convinces us that an equation like $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ is far more interesting than the rather dull statement that $3 = 3$. A comparable remark applies if we go up in dimension: equivalent categories are thought to be essentially equal, but an equivalence is of greater interest if the involved categories look different. Numerous examples of equivalences of "different" categories relate a category X and the dual of a category A. Such an equivalence is called a ***dual equivalence*** or simply a ***duality***, and is usually denoted by $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$. Like for every other equivalence, a duality allows us to transport properties from one side to the other. The presence of the dual category on one side is often useful since our knowledge of properties of a category is typically asymmetric. Indeed, many "everyday categories" admit a representable and hence limit preserving functor to Set. Therefore in these categories limits are "easy"; however, colimits are often "hard". In these circumstances, an equivalence $\mathsf{X} \simeq \mathsf{A}^{\mathrm{op}}$ together with the knowledge of limits in A help us understand colimits in X. The dual situation, where colimits are "easy" and limits are "hard", frequently emerges in the context of coalgebras. For example, the category $\mathrm{CoAlg}(V)$ of coalgebras for the Vietoris functor V on the category BooSp of Boolean spaces and continuous functions is known to be equivalent to the dual of the category BAO with objects Boolean algebras B with an operator $h : B \to B$ satisfying the equations

$$h(\bot) = \bot \qquad \text{and} \qquad h(x \vee y) = h(x) \vee h(y),$$

and with morphisms the Boolean homomorphisms which also preserve the additional unary operation (see [20]). It is a trivial observation that BAO is a category of algebras over Set defined by a (finite) set of operations and a collection of equations; every such category is known to be complete and cocomplete. Notably, the equivalence $\mathrm{CoAlg}(V) \simeq$