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Highest weight theory for finite-dimensional graded algebras with triangular decomposition

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ABSTRACT

We show that the category of graded modules over a finite-dimensional graded algebra admitting a triangular decomposition can be endowed with the structure of a highest weight category. When the algebra is self-injective, we show furthermore that this highest weight category has tilting modules in the sense of Ringel. This provides a new perspective on the representation theory of such algebras, and leads to several new structures attached to them. There are a wide variety of examples in algebraic Lie theory to which this applies: restricted enveloping algebras, Lusztig's small quantum groups, hyperalgebras, finite quantum groups, and restricted rational Cherednik algebras.

Contents

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1. Introduction

The goal of this paper, and its sequel [11], is to develop new structures in the representation theory of a class of algebras commonly encountered in algebraic Lie theory: finite-dimensional $\mathbb{Z}$-graded algebras A which admit a *triangular decomposition*, i.e., a vector space decomposition

$$A^- \otimes T \otimes A^+ \xrightarrow{\sim} A \tag{1}$$

into graded subalgebras given by the multiplication map, where we assume that A^- is concentrated in negative degree, T in degree zero, and A^+ in positive degree.

There are a variety of examples:

(1) Restricted enveloping algebras $\overline{U}(\mathfrak{g}_K)$;
(2) Lusztig's small quantum groups $\mathbf{u}_\zeta(\mathfrak{g})$, at a root of unity ζ;
(3) Hyperalgebras $\mathfrak{u}_r(\mathfrak{g}) := \mathrm{Dist}(G_r)$ on the Frobenius kernel G_r;
(4) Finite quantum groups $\mathcal{D}$ associated to a finite group G;
(5) Restricted rational Cherednik algebras (RRCAs) $\overline{\mathsf{H}}_{\mathbf{c}}(W)$ at $t = 0$;
(6) The center of smooth blocks of RRCAs at $t = 0$;
(7) RRCAs $\overline{\mathsf{H}}_{1,\mathbf{c}}(W)$ at $t = 1$ in positive characteristic.

There are many more examples, but the above examples are the ones we will address in more detail in Section 8. The representation theory of these algebras has important applications to other areas of mathematics. For instance, to symplectic algebraic geometry [30,5,7,9], to algebraic combinatorics [34,47], and to algebraic groups in positive characteristic [39,2]. The applications mostly derive in one way or another from computing the graded character of irreducible modules.

If we look at the list above, we can say that examples 1 to 3 share a "common background", as do examples 5 to 7, but taken in their totality, the algebras do not have much in common—except that they all admit a triangular decomposition. On the other hand, their representation theory behaves in a remarkably uniform way. This suggests that it is worthwhile developing a systematic approach to the representation theory of algebras with triangular decomposition.