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Variation of the Nazarov–Sodin constant for random plane waves and arithmetic random waves

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ABSTRACT

This is a manuscript containing the full proofs of results announced in [10], together with some recent updates. We prove that the *Nazarov–Sodin constant*, which up to a natural scaling gives the leading order growth for the expected number of nodal components of a random Gaussian field, genuinely depends on the field. We then infer the same for "arithmetic random waves", i.e. random toral Laplace eigenfunctions.

1. Introduction

For $m \geq 2$, let $f : \mathbb{R}^m \to \mathbb{R}$ be a *stationary* centred Gaussian random field, and $r_f : \mathbb{R}^m \to \mathbb{R}$ its covariance function

$$r_f(x) = \mathbb{E}[f(0) \cdot f(x)] = \mathbb{E}[f(y) \cdot f(x + y)].$$

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Given such an f, let ρ denote its spectral measure, i.e. the Fourier transform of r_f (assumed to be a probability measure); note that prescribing ρ uniquely defines f by Kolmogorov's Theorem (cf. [6, Chapter 3.3]). In what follows we often allow for ρ to vary; it will be convenient to let f_ρ denote a random field with spectral measure ρ. We further assume that a.s. f_ρ is sufficiently smooth, and that the distribution of f and its derivatives is non-degenerate in an appropriate sense (a condition on the support of ρ).

1.1. Nodal components and the Nazarov–Sodin constant

Let $\mathcal{N}(f_\rho; R)$ be the number of connected components of $f_\rho^{-1}(0)$ in $B_0(R)$ (the radius-R ball centred at 0), usually referred to as the *nodal components* of f_ρ; $\mathcal{N}(f_\rho; R)$ is a random variable. Assuming further that f_ρ is ergodic (equivalently, ρ has no atoms), with non-degenerate gradient distribution (equivalent to ρ not being supported on a hyperplane passing through the origin), Nazarov and Sodin [21, Theorem 1] evaluated the expected number of nodal components of f_ρ to be asymptotic to

$$\mathbb{E}[\mathcal{N}(f_\rho; R)] = c_{NS}(\rho) \cdot \mathrm{Vol}(B(1)) \cdot R^m + o(R^m), \tag{1.1}$$

where $c_{NS}(\rho) \geq 0$ is a constant, subsequently referred to as the "Nazarov–Sodin constant" of f_ρ, and $\mathrm{Vol}(B(1))$ is the volume of the radius-1 m-ball $B(1) \subseteq \mathbb{R}^m$. They also established convergence in mean, i.e., that

$$\mathbb{E}\left[\left|\frac{\mathcal{N}(f_\rho; R)}{\mathrm{Vol}(B(1)) \cdot R^m} - c_{NS}(\rho)\right|\right] \to 0; \tag{1.2}$$

it is a consequence of the assumed ergodicity of the underlying random field f_ρ. In this manuscript we will consider c_{NS} as a function of the spectral density ρ, without assuming that f_ρ is ergodic. To our best knowledge, the value of $c_{NS}(\rho)$, even for a single ρ, was not rigorously known heretofore.

For $m = 2$, $\rho = \rho_{\mathcal{S}^1}$ the uniform measure on the unit circle $\mathcal{S}^1 \subseteq \mathbb{R}^2$ (i.e. $d\rho = \frac{d\theta}{2\pi}$ on $\mathcal{S}^1$, and vanishing outside the circle) the corresponding random field f_{RWM} is known as *random monochromatic wave*; according to Berry's *Random Wave Model* [3], f_{RWM} serves as a universal model for Laplace eigenfunctions on generic surfaces in the high energy limit. The corresponding *universal* Nazarov–Sodin constant

$$c_{\mathrm{RWM}} = c_{NS}\left(\frac{d\theta}{2\pi}\right) > 0 \tag{1.3}$$

was proven to be strictly positive [14]. Already in [4], Bogomolny and Schmit employed the percolation theory to predict its value, but recent numerics by Nastacescu [13], Konrad [8] and Beliaev–Kereta [2], consistently indicate a 4.5–6% deviation from these predictions.

More generally, let $(\mathcal{M}^m, g)$ be a smooth compact Riemannian manifold of volume $\mathrm{Vol}(\mathcal{M})$. Here the restriction of a fixed random field $f : \mathcal{M} \to \mathbb{R}$ to growing domains, as