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Geometry of Uhlenbeck partial compactification of orthogonal instanton spaces and the K-theoretic Nekrasov partition functions

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Let $\mathcal{M}_n^K$ be the moduli space of framed K-instantons over S^4 with instanton number n when K is a compact simple Lie group of classical type. Let $\mathcal{U}_n^K$ be the Uhlenbeck partial compactification of $\mathcal{M}_n^K$. A scheme structure on $\mathcal{U}_n^K$ is endowed by Donaldson as an algebro-geometric Hamiltonian reduction of ADHM data. In this paper, for $K = \mathrm{SO}(N, \mathbb{R})$, $N \geq 5$, we prove that $\mathcal{U}_n^K$ is an irreducible normal variety with smooth locus $\mathcal{M}_n^K$. Hence, together with the author's previous results on $\mathrm{USp}(N)$, the K-theoretic Nekrasov partition function for any simple classical group other than $\mathrm{SO}(3, \mathbb{R})$, is interpreted as a generating function of Hilbert series of the instanton moduli spaces.

Using this approach we also study the case $K = \mathrm{SO}(4, \mathbb{R})$ which is the unique semisimple but non-simple classical group.

Contents

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1. Introduction

Let K be a compact classical group. The K-theoretic Nekrasov partition function is defined by Nekrasov and Shadchin as the formal sum of the equivariant integrations of K-theory classes on the ADHM quiver representation space associated to K-instantons [38]. Our previous result [11] says that it is the generating function of the Hilbert series of the coordinate rings of the framed K-instanton moduli spaces $\mathcal{M}_n^K$ over all instanton numbers n if $K = \mathrm{USp}(N/2)$ the real symplectic group where $N \in 2\mathbb{Z}_{\geq 0}$. We aim to prove the parallel result for $K = \mathrm{SO}(N, \mathbb{R})$. Note that such a result has been known for $K = \mathrm{SU}(N)$ essentially due to Crawley-Boevey [14].

1.1. Main result

We state the main result of the paper in this subsection. For the purpose we need to describe the ADHM data of instantons following Donaldson's argument [15]. First we consider the vector space $\mathbf{M}$ of ordinary ADHM quiver representations coming from framed $\mathrm{SU}(N)$-instantons with instanton number k. It is given as $\mathbf{M} = \mathrm{End}(V)^{\oplus 2} \oplus \mathrm{Hom}(W, V) \oplus \mathrm{Hom}(V, W)$. $\mathbf{M}$ is a cotangent space, hence naturally a symplectic vector space. The adjoint $\mathrm{GL}(V)$-action preserves the symplectic structure.

We fix an instanton number n. Let $k = 2n, 4n$ or n according to $K = \mathrm{SO}(N, \mathbb{R})$ ($N \geq 4$), $\mathrm{SO}(3, \mathbb{R})$ or $\mathrm{USp}(N/2)$. Let

$$G := \begin{cases} \mathrm{Sp}(n) & \text{if } K = \mathrm{SO}(N, \mathbb{R}) \text{ with } N \geq 4 \\ \mathrm{Sp}(2n) & \text{if } K = \mathrm{SO}(3, \mathbb{R}) \\ \mathrm{O}(n) & \text{if } K = \mathrm{USp}(N/2) \text{ with } N \in 2\mathbb{Z}_{\geq 0}. \end{cases}$$

Let V, W be the vector representations of $G, K_{\mathbb{C}}$ respectively.

We define a symplectic subspace of $\mathbf{M}$ as

$$\mathbf{N} := \mathbf{N}_{V,W} := \{(B_1, B_2, i, j) \in \mathbf{M} \mid B_1 = B_1^*, B_2 = B_2^*, j = i^*\}$$