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Algebraic differential equations from covering maps

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ABSTRACT

Let Y be a complex algebraic variety, $G \curvearrowright Y$ an action of an algebraic group on Y, $U \subseteq Y(\mathbb{C})$ a complex submanifold, $\Gamma < G(\mathbb{C})$ a discrete, Zariski dense subgroup of $G(\mathbb{C})$ which preserves U, and $\pi : U \to X(\mathbb{C})$ an analytic covering map of the complex algebraic variety X expressing $X(\mathbb{C})$ as $\Gamma\backslash U$. We note that the theory of elimination of imaginaries in differentially closed fields produces a generalized Schwarzian derivative $\widetilde{\chi} : Y \to Z$ (where Z is some algebraic variety) expressing the quotient of Y by the action of the constant points of G. Under the additional hypothesis that the restriction of π to some set containing a fundamental domain is definable in an o-minimal expansion of the real field, we show as a consequence of the Peterzil–Starchenko o-minimal GAGA theorem that the *prima facie* differentially analytic relation $\chi := \widetilde{\chi} \circ \pi^{-1}$ is a well-defined, differential constructible function. The function χ nearly inverts π in the sense that for any differential field K of meromorphic functions, if $a, b \in X(K)$ then $\chi(a) = \chi(b)$ if and only if after suitable restriction there is some $\gamma \in G(\mathbb{C})$ with $\pi(\gamma \cdot \pi^{-1}(a)) = b$.

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1. Introduction

As is well-known, the complex exponential function $\exp : \mathbb{C} \to \mathbb{C}^\times$ admits a local analytic inverse, the logarithm function, but the logarithm cannot be made into a globally defined analytic function. The ambiguity in the choice of a branch of the logarithm comes from addition by an element of the discrete group $2\pi i\mathbb{Z}$. Hence, if we regard the logarithm as acting on meromorphic function via $f \mapsto \log \circ f$, then while the operator $f \mapsto \log(f)$ is not well-defined, the logarithmic derivative, $f \mapsto \frac{d}{dz}(\log(f))$ is. Of course, more is true in that the logarithmic derivative is given by the simple differential algebraic formula $\frac{d}{dz}(\log(f)) = \frac{f'}{f}$.

There are many ways to see that the logarithmic derivative is a differential rational function. This fact follows from a direct computation, or from Kolchin's general theory of logarithmic differentiation on algebraic groups [12], or from the techniques we employ in this paper. In Section 5 we discuss the algebraic construction of the logarithmic derivative in detail.

The purpose of this paper is to show that under very general hypotheses, differential analytic operators constructed by inverting analytic covering maps and then applying differential operators to kill the action of the constant points of some algebraic group are actually differential algebraic. Let us describe more precisely what we have in mind. We are given an algebraic group G over $\mathbb{C}$, a complex algebraic variety Y, a regular action of G on Y, a complex submanifold $U \subseteq Y(\mathbb{C})$ of $Y(\mathbb{C})$, a Zariski dense subgroup $\Gamma < G(\mathbb{C})$ for which Γ preserves U and an analytic covering map $\pi : U \to X(\mathbb{C})$ expressing the complex algebraic variety X as the quotient $\Gamma\backslash U$. Because π is a covering map, the inverse $\pi^{-1} : X(\mathbb{C}) \to U$ is a many-valued analytic function, well-defined up to the action of Γ. Using the theory of elimination of imaginaries in differential fields, we produce a differential constructible function, which we call a generalized Schwarzian associated to the action of G on Y, $\widetilde{\chi} : Y \to Z$ (where Z is an algebraic variety) so that for any differential field M having field of constants $\mathbb{C}$ and points $a, b \in Y(M)$ one has $\widetilde{\chi}(a) = \widetilde{\chi}(b)$ if and only if there is some $\gamma \in G(\mathbb{C})$ with $\gamma \cdot a = b$. Hence, the differential analytic operator $\chi := \widetilde{\chi} \circ \pi^{-1}$ gives a well defined function $X(M) \to Z(M)$ for M any field of meromorphic functions.

Under a mild hypothesis on π, namely that there is an o-minimal expansion of $\mathbb{R}$ in which there is a definable subset $F \subseteq Y(\mathbb{C})$ for which the restriction of π is definable and surjective onto $X(\mathbb{C})$, we then deduce from a remarkable theorem of Peterzil–Starchenko that the *a priori* differential *analytically* constructible function χ is in fact differential *algebraically* constructible. That is, the map χ may be expressed piecewise as a rational function of its argument and some its derivatives and pieces in the decomposition of its domain are defined by finitely many algebraic differential equations and inequations. The definability hypothesis holds in many cases of interest, such as for the covering maps associated to moduli spaces of abelian varieties and of the universal families of abelian varieties over these moduli spaces.