Advances in Mathematics 330 (2018) 1101–1159

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Free functional inequalities on the circle ☆

Ionel Popescu [a,b,c,*]

[a] School of Mathematics, Georgia Institute of Technology, Atlanta, GA, 30332, United States
[b] IMAR, 21 Calea Grivitei Street, 010702 Bucharest, Romania
[c] FMI, Str. Academiei nr. 14, sector 1, 010014, Bucharest, Romania

ARTICLE INFO

*Article history:*
Received 17 November 2015
Received in revised form 16 August 2017
Accepted 16 October 2017
Available online xxxx
Communicated by Dan Voiculescu

*Keywords:*
Free probability
Functional inequalities
Mass transport

ABSTRACT

In this paper we deal with free functional inequalities on the circle. There are some interesting changes from their classical counterparts. For example, the free Poincaré inequality has a slight change which seems to account for the lack of invariance under rotations of the base measure. Another instance is the modified Wasserstein distance on the circle which provides the tools for analyzing transportation, Log-Sobolev, and HWI inequalities.
These new phenomena also indicate that they have classical counterparts, which does not seem to have been investigated thus far.



☆ This work was partially supported by Simon Collaboration grant no. 318929 and UEFISCDI grant PN-III-P4-ID-PCE-2016-0372.

* Correspondence to: School of Mathematics, Georgia Institute of Technology, Atlanta, GA, 30332, United States.
*E-mail address:* ipopescu@math.gatech.edu.

https://doi.org/10.1016/j.aim.2017.10.021
0001-8708/

## 1. Introduction

An intensive area of research nowadays is functional inequalities. In the classical case these reflect various aspects of operator theory, mass transport, concentration of measure, isoperimetry and analysis of Markov processes.

To describe the setup, we start with a Riemannian manifold $M$ and a probability measure $\nu$ on it. The classical transportation cost inequality states that for some $\rho > 0$ and any other choice of probability measure $\mu$ on $M$ the following holds

$$\rho W_2^2(\mu, \nu) \leq E(\mu|\nu). \tag{$T(\rho)$}$$

Here $W_2(\mu, \nu)$ denotes the Wasserstein distance between $\mu$ and $\nu$ (probability measures of finite second moment) given by

$$W_2(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left( \iint d(x, y)^2 \pi(dx, dy) \right)^{1/2}$$

where $\Pi(\mu, \nu)$ stands for the set of probability measures on $M \times M$ with marginals $\mu$ and $\nu$ and $d(x, y)$ is the geodesic distance between the points $x, y$. Also we use here

$$E(\mu|\nu) = \int \log \frac{d\mu}{d\nu}\, d\mu$$

for the relative entropy of $\mu$ with respect to $\nu$ if $\mu << \nu$ and $+\infty$ otherwise.

The classical Log-Sobolev inequality states that for any $\mu$

$$E(\mu|\nu) \leq \frac{1}{2\rho} I(\mu|\nu) \tag{$LSI(\rho)$}$$

where

$$I(\mu|\nu) = \int \left| \nabla \log \frac{d\mu}{d\nu} \right|^2 d\mu$$

is the Fisher information of $\mu$ with respect to $\nu$ defined in the case $\mu << \nu$ and $\frac{d\mu}{d\nu}$ being differentiable.

Another more refined inequality is the HWI inequality introduced in [21] which states that for any choice of the measure $\mu$,

$$E(\mu|\nu) \leq \sqrt{I(\mu|\nu)}\, W_2(\mu, \nu) - \frac{\rho}{2} W_2^2(\mu, \nu). \tag{$HWI(\rho)$}$$

Finally, the classical Poincaré inequality states that for any compactly supported and smooth function $\psi$ on $M$,

$$\rho \operatorname{Var}_\nu(\psi) \leq \int |\nabla\psi|^2 \nu(dx) \tag{$P(\rho)$}$$