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Tensor product of modules over a vertex algebra

José I. Liberati

Ciem - CONICET, Medina Allende y Haya de la Torre, Ciudad Universitaria, (5000) Córdoba, Argentina

A R T I C L E   I N F O

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A B S T R A C T

We find a necessary and sufficient condition for the existence of the tensor product of modules over a vertex algebra. We define the notion of vertex bilinear map and provide two algebraic constructions of the tensor product, where one of them is of ring theoretical type. We show the relation between tensor product and vertex homomorphisms. We prove commutativity of the tensor product. We also prove associativity of the tensor product of modules under certain necessary and sufficient condition. Finally, we show certain functorial properties of vertex homomorphism and the tensor product.

1. Introduction

This is the first of a series of papers (the second one is [24]) trying to extend certain restricted definitions and constructions already developed for vertex operator algebras to the general framework of vertex algebras without assuming any grading condition neither on the vertex algebra nor on the modules involved, and putting a strong emphasis on the commutative associative algebra point of view instead of the Lie theoretical point of view. These series of works help eliminate the Jacobi identity from the theory of vertex

E-mail address: joseliberati@gmail.com.

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algebras, by considering the correct point of view based on the theory of commutative associative algebras.

The study of tensor product theory for representations of a chiral algebra (or vertex operator algebra), was initiated by physicists ([27] and references therein). Later, Gaberdiel [7] pointed out that Borcherds suggested a ring like construction, and he developed it on a physical level of rigor. Simultaneously, Huang and Lepowsky [9–13] and [8], gave the first mathematically rigorous notion and construction of the tensor product of certain graded modules over certain vertex operator algebras, by using analytical methods. In [21], Li gave a formal variable approach to the tensor product theory, following classical Lie algebra theory and restricting the definitions and constructions to certain graded modules over a rational vertex operator algebra.

Many mathematicians believe that there should be an algebraic and ring type construction of the tensor product of modules over a vertex algebra. It has been a necessary and missing part of the theory in the last 30 years. The present work tries to fill this gap.

In order to have a purely algebraic definition and construction, we loose the analytical and geometric interpretation, but we simplify the theory.

Our work is strongly influenced by the works of Haisheng Li, especially his thesis. Most of the ideas and tricks used in this work are taken from his papers, but applied in a different context. The ideas and constructions are very simple and natural.

In section 2, we present the basic definitions and notations. In section 3, we find a necessary and sufficient condition for the existence of the tensor product of modules over a vertex algebra, that we called the *kernel intertwining operator full equality condition*, and we present the first construction.

Since we have a different point of view of very well known notions, in section 4 we define the notion of vertex bilinear map that replaces the notion of intertwining operator. This allows us to redefine the notion of tensor product with a ring theoretical interpretation (Definition 4.4), and provide the second algebraic construction of the tensor product, which is of ring theoretical type.

In section 5, we introduce the notion of (right) vertex homomorphism, producing the "Hom" functor for modules over a vertex algebra, obtaining (from our point of view) an analogous of the Hom functor for modules over an associative commutative algebra, and we denote it as $Vhom$. We prefer to think about it in this way, instead of the Hom functor in Lie theory. In the first part of this section, we follow [21], [22] and [4]. The most important functors in homological algebra are Hom, tensor product and functors derived from them. In the case of vertex algebras, we should consider $Vhom$. We shall show that there is an intimate relationship between $Vhom$ and tensor product, they form an adjoint pair of functors (see Theorem 5.6, Theorem 5.14 and Corollary 5.15).

In section 6, we prove the commutativity of the tensor product. Then we prove its associativity under certain (algebraic and natural) necessary and sufficient condition. In order to prove associativity, we shall use the universal property of the tensor product