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L_p dual curvature measures ☆

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A B S T R A C T

A new family of geometric Borel measures on the unit sphere is introduced. Special cases include the L_p surface area measures (which extend the classical surface area measure of Aleksandrov and Fenchel & Jessen) and L_p-integral curvature (which extends Alkesandrov's integral curvature) in the L_p Brunn–Minkowski theory. It also includes the dual curvature measures (which are the duals of Federer's curvature measures) in the dual Brunn–Minkowski theory. This partially unifies the classical theory of mixed volumes and the newer theory of dual mixed volumes.

1. Introduction

Surface area measure and integral curvature are two important geometric measures of convex bodies in the Euclidean n-space, $\mathbb{R}^n$. Integral curvature measures the images of the

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Gauss map of a convex body, while surface area measure measures the inverse images of the Gauss map of a convex body. Both measures are fundamental concepts in the classical Brunn–Minkowski theory of convex bodies. The Minkowski problem characterizing surface area measure and the Aleksandrov problem characterizing integral curvature are two well-known problems. In modern convex geometry, the L_p Brunn–Minkowski theory and the dual Brunn–Minkowski theory generalize and dualize the classical Brunn–Minkowski theory. The L_p surface area measures were introduced in [31], and L_p integral curvatures were recently defined in [19]. Equally fundamental geometric measures in the dual Brunn–Minkowski theory were only constructed very recently in [18]. They are called dual curvature measures (and are dual to Federer's curvature measures). Minkowski problems associated with these geometric measures are major problems in convex geometric analysis, which are far from being completely solved.

The purpose of this paper is to continue the study begun in [18] and to construct L_p dual curvature measures. It turns out that the L_p surface area measure, L_p integral curvatures, and dual curvature measures are all special cases of the now-to-be introduced L_p dual curvature measures. These lead to a unified concept of mixed volume that includes Minkowski's classical first mixed volume, L_p mixed volumes, L_p entropy, as well as dual mixed volumes as special cases. We shall demonstrate a surprising connection between the L_p Brunn–Minkowski theory and the dual Brunn–Minkowski theory by establishing geometric inequalities and variational integral formulas for the unified mixed volumes and for the new L_p dual curvature measures. We pose the L_p dual Minkowski problem for L_p dual curvature measure which opens a new direction of study in convex geometric analysis. Detailed explanations are provided below.

The L_p Brunn–Minkowski theory as a generalization of the classical Brunn–Minkowski theory has attracted increasing interest in recent years partly due to its wide range of connections with other subjects such as affine geometry, Banach space theory, harmonic analysis, and partial differential equations. The core concept in the L_p Brunn–Minkowski theory (introduced in [31]) is the notion of the L_p *surface area measure* which is a Borel measure (defined on the unit sphere) for each convex body in $\mathbb{R}^n$ that contains the origin in its interior. The L_p-cosine transform (a spherical variant of the Fourier transform) of the L_p surface area measure turns out to yield a finite dimensional Banach norm. The associated affine isoperimetric inequality for the volume of the unit ball of this Banach norm is known as the L_p *Petty projection inequality* which was established in [32] and is a profound strengthening of the classical isoperimetric inequality. The Radon–Nikodym derivative of the L_p surface area measure with respect to the spherical Lebesgue measure is called the L_p *curvature* function. The integral of the L_p curvature function (raised to an appropriate power) over the unit sphere is L_p *affine surface area* which has been a focus of study in affine geometry and valuation theory, see e.g. [14,28,35,42,45,46]. Finding the necessary and sufficient conditions for a given measure to guarantee that it is the L_p surface area measure is the existence problem called the L_p *Minkowski problem* posed in [31]. Solving the L_p Minkowski problem requires solving a degenerate and singular Monge–Ampère type equation on the unit sphere. The problem has been