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Orthogonal free quantum group factors are strongly 1-bounded

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A B S T R A C T

We prove that the orthogonal free quantum group factors $\mathcal{L}(\mathbb{F}O_N)$ are strongly 1-bounded in the sense of Jung. In particular, they are not isomorphic to free group factors. This result is obtained by establishing a spectral regularity result for the edge reversing operator on the quantum Cayley tree associated to $\mathbb{F}O_N$, and combining this result with a recent free entropy dimension rank theorem of Jung and Shlyakhtenko.

1. Introduction

The theory of discrete quantum groups provides a rich source of interesting examples of C*-algebras and von Neumann algebras. In addition to ordinary discrete groups, there is a wealth of examples and phenomena arising from genuinely quantum groups [15,42,7,29,25,1]. Within the class of non-amenable discrete quantum groups, the so-called *free*

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quantum groups of Wang and Van Daele [34,41] somehow form the most prominent examples.

In this paper, our main focus is on the structural theory of a family of II_1-factors associated to a special family of free quantum groups, called the *orthogonal free quantum groups*. Given an integer $N \geq 2$, the orthogonal free quantum group $\mathbb{F}O_N$ is the discrete quantum group defined via the full Woronowicz C*-algebra

$$C^*_{\mathrm{f}}(\mathbb{F}O_N) = \langle u_{ij}, 1 \leq i, j \leq N \mid u = [u_{ij}] \text{ unitary}, \ u_{ij} = u^*_{ij} \ \forall i, j \rangle.$$

The C*-algebra $C^*_{\mathrm{f}}(\mathbb{F}O_N)$ can be interpreted simultaneously as a free analogue of the C*-algebra of continuous functions on the real orthogonal group O_N, and also as a "matricial" analogue of the full free group C*-algebra $C^*_{\mathrm{f}}((\mathbb{Z}/2\mathbb{Z})^{\star N})$. Indeed, by quotienting by the commutator ideal or by setting $u_{ij} = 0$ $(i \neq j)$, respectively, we obtain surjective Woronowicz-C*-morphisms

$$C^*_{\mathrm{f}}(\mathbb{F}O_N) \to C(O_N), \qquad C^*_{\mathrm{f}}(\mathbb{F}O_N) \to C^*_{\mathrm{f}}((\mathbb{Z}/2\mathbb{Z})^{\star N}).$$

Using the (tracial) Haar state $h : C^*_{\mathrm{f}}(\mathbb{F}O_N) \to \mathbb{C}$, the GNS construction yields in the usual way a Hilbert space $\ell^2(\mathbb{F}O_N)$ and a corresponding von Neumann algebra $\mathcal{L}(\mathbb{F}O_N) = \pi_h(C^*_{\mathrm{f}}(\mathbb{F}O_N))'' \subseteq B(\ell^2(\mathbb{F}O_N))$, where π_h denotes the GNS representation. Over the past two decades, the structure of the algebras $\mathcal{L}(\mathbb{F}O_N)$ has been investigated by many hands, and in many respects $\mathbb{F}O_N$ and $\mathcal{L}(\mathbb{F}O_N)$ $(N \geq 3)$ were shown to share many properties with free groups F_n and their von Neumann algebras $\mathcal{L}(F_n)$.

For example, $\mathcal{L}(\mathbb{F}O_N)$ is a full type II_1-factor, it is strongly solid, and in particular prime and has no Cartan subalgebra; it has the Haagerup property (HAP), is weakly amenable with Cowling–Haagerup constant 1 (CMAP), and satisfies the Connes' Embedding conjecture [3,32,19,9,17,11,16]. Moreover, it is known that $\mathcal{L}(\mathbb{F}O_N)$ behaves asymptotically like a free group factor in the sense that the canonical generators of $\mathcal{L}(\mathbb{F}O_N)$ become strongly asymptotically free semicircular systems as $N \to \infty$ [5,10].

With these many similarities between $\mathcal{L}(\mathbb{F}O_N)$ and $\mathcal{L}(F_n)$ at hand, the following question naturally arises:

Can $\mathcal{L}(\mathbb{F}O_N)$ be isomorphic to a free group factor?

This particular question has been circulating within the operator algebra and quantum group communities ever since the publication of Banica's thesis [3,4] in the mid 1990's, which first connected the corepresentation theory of free quantum groups to Voiculescu's free probability theory. This deep connection with free independence established by Banica was a direct inspiration for the many structural results for $\mathcal{L}(\mathbb{F}O_N)$ described in the previous paragraph. In this paper, our main objective is to finally answer the above question in the negative.

The first evidence suggesting a negative answer to an isomorphism with a free group factor came from the work of the second author [36], where the L^2-cohomology of