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Nevanlinna theory of the Askey–Wilson divided difference operator

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A B S T R A C T

This paper establishes a version of Nevanlinna theory based on Askey–Wilson divided difference operator for meromorphic functions of finite logarithmic order in the complex plane $\mathbb{C}$. A second main theorem that we have derived allows us to define an Askey–Wilson type Nevanlinna deficiency which gives a new interpretation that one should regard many important infinite products arising from the study of basic hypergeometric series as zero/pole-scarce. That is, their zeros/poles are indeed deficient in the sense of difference Nevanlinna theory. A natural consequence is a version of Askey–Wilson type Picard theorem. We also give an alternative and self-contained characterisation of the kernel functions of the Askey–Wilson operator. In addition we have established a version of unicity theorem in the sense of Askey–Wilson. This paper concludes with an application to difference

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equations generalising the Askey–Wilson second-order divided difference equation.

1. Introduction

Without loss of generality, we assume q to be a complex number with $|q| < 1$. Askey and Wilson evaluated a q-beta integral ([6, Theorem 2.1]) that allows them to construct a family of orthogonal polynomials ([6, Theorems 2.2–2.5]) which are eigen-solutions of a second order difference equation ([6, §5]) now bears their names. The divided difference operator $\mathcal{D}_q$ that appears in the second-order difference equation is called *Askey–Wilson operator*. These polynomials, their orthogonality weight, the difference operator and related topics have found numerous applications and connections with a wide range of research areas beyond the basic hypergeometric series. These research areas include, for examples, Fourier analysis ([11]), interpolations ([38], [31]), combinatorics ([20]), Markov process ([12], [44]), quantum groups ([34], [42]), double affine Hecke (Cherednik) algebras ([15], [33]).

In this paper, we show, building on the strengths of the work of Halburd and Korhonen [23], [24] and as well as our earlier work on logarithmic difference estimates ([17], [18]), that there is a very natural function theoretic interpretation of the Askey–Wilson operator (abbreviated as AW-operator) $\mathcal{D}_q$ and related topics. It is not difficult to show that the AW-operator is well-defined on meromorphic functions. In particular, we show that there is a Picard theorem associates with the Askey–Wilson operator just as the classical Picard theorem is associated with the conventional differential operator f'. Moreover, we have obtained a full-fledged Nevanlinna theory for slow-growing meromorphic functions with respect to the AW-operator on $\mathbb{C}$ for which the associated Picard theorem follows as a special case, just as the classical Picard theorem is a simple consequence of the classical Nevanlinna theory ([40], see also [41], [27] and [46]). This approach allows us to gain new insights into the $\mathcal{D}_q$ and that give a radically different viewpoint from the established views on the value distribution properties of certain meromorphic functions, such as the Jacobi theta-functions, generating functions of certain orthogonal polynomials that were used in L. J. Rogers' derivation of the two famous Rogers–Ramanujan identities [43], etc. We also characterise the functions that lie in the kernel of the Askey–Wilson operator, which we can regard as the *constants* with respect to the AW-operator.

A value a which is not assumed by a meromorphic function f is called a *Picard (exceptional) value*. The Picard theorem states that if a meromorphic f that has three Picard values, then f necessarily reduces to a constant. For each complex number a, Nevanlinna defines a deficiency $0 \le \delta(a) \le 1$. If $\delta(a) \sim 1$, then that means f rarely assumes a. In fact, if a is a Picard value of f, then $\delta(a) = 1$. If f assumes a frequently, then $\delta(a) \sim 0$. Nevanlinna's second fundamental theorem implies that $\sum_{a\in\mathbb{C}} \delta(a) \le 2$ for