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New dimension spectra: Finer information on scaling and homogeneity ☆

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A B S T R A C T

We introduce a new dimension spectrum motivated by the Assouad dimension; a familiar notion of dimension which, for a given metric space, returns the minimal exponent $\alpha \geqslant 0$ such that for any pair of scales $0 < r < R$, any ball of radius R may be covered by a constant times $(R/r)^\alpha$ balls of radius r. To each $\theta \in (0, 1)$, we associate the appropriate analogue of the Assouad dimension with the restriction that the two scales r and R used in the definition satisfy $\log R/\log r = \theta$. The resulting 'dimension spectrum' (as a function of θ) thus gives finer geometric information regarding the scaling structure of the space and, in some precise sense, interpolates between the upper box dimension and the Assouad dimension. This latter point is particularly useful because the spectrum is generally better behaved than the Assouad dimension. We also consider the corresponding 'lower spectrum', motivated by the lower dimension, which acts as a dual to the Assouad spectrum.

We conduct a detailed study of these dimension spectra; including analytic, geometric, and measureability properties. We also compute the spectra explicitly for some common examples of fractals including decreasing sequences with decreasing gaps and spirals with sub-exponential and monotonic winding. We also give several applications of our results, in-

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cluding: dimension distortion estimates under bi-Hölder maps for Assouad dimension and the provision of new bi-Lipschitz invariants.

Contents

1. New dimension spectra, summary of results, and organisation of paper

The Assouad dimension is a fundamental notion of dimension used to study fractal objects in a wide variety of contexts. It was popularised by Assouad in the 1970s [2, 3] and subsequently took on significant importance in embedding theory. Recall the famous *Assouad Embedding Theorem* which states that if (X, d) is a metric space with the doubling property (equivalently, with finite Assouad dimension), then (X, d^ε) admits a bi-Lipschitz embedding into some finite dimensional Euclidean space for any $\varepsilon \in (0, 1)$. The notion we now call Assouad dimension does go back further, however, to Larman's work in the 1960s [28,29] and even to Bouligand's 1928 paper [4]. It is also worth noting that, due to its deep connections with tangents (see [37]), it is intimately related to pioneering work of Furstenberg on micro-sets which goes back to the 1960s, see [18]. Roughly speaking, the Assouad dimension assigns a number to a given metric space which quantifies the most difficult location and scale at which to cover the space. More precisely, it considers two scales $0 < r < R$ and finds the maximal exponential growth rate of $N(B(x, R), r)$ as R and r decrease, where $N(E, r)$ is the minimal number of r-balls required to cover a set E.

The Assouad dimension has found important applications in a wide variety of contexts, including a sustained importance in embedding theory, see [46,42,43]. It is also central to quasi-conformal geometry, see [20,48,37], and has recently been gaining significant attention in the literature on fractal geometry and geometric measure theory, see for example [36,35,12,13,23,24,31]. However, its application and interest does not end there. For example, in the study of fractional Hardy inequalities, if the boundary of a domain in $\mathbb{R}^d$ has Assouad dimension less than or equal to $d-p$, then the domain admits the fractional p-Hardy inequality, see [1,27,32]. Also, Hieronymi and Miller have recently