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Continuity and representation of valuations on star bodies

Pedro Tradacete [a,*,1], Ignacio Villanueva [b,2]

[a] *Mathematics Department, Universidad Carlos III de Madrid, 28911 Leganés (Madrid), Spain*

[b] *Departamento de Análisis Matemático & Instituto de Matemática Interdisciplinar, Facultad de Matemáticas, Universidad Complutense de Madrid, Madrid 28040, Spain*

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ABSTRACT

It is shown that every continuous valuation defined on n-dimensional star bodies has an integral representation in terms of the radial function. Our argument is based on the non-trivial fact that continuous valuations are uniformly continuous on bounded sets. We also characterize continuous valuations on the n-dimensional star bodies that arise as restriction of a measure on $\mathbb{R}^n$.

* Corresponding author.
E-mail addresses: ptradace@math.uc3m.es (P. Tradacete), ignaciov@mat.ucm.es (I. Villanueva).

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1. Introduction

A valuation is a function V, defined on a given class of sets $\mathcal{F}$, which satisfies that, for every $A, B \in \mathcal{F}$ such that $A \cup B$, $A \cap B \in \mathcal{F}$, one has

$$V(A \cup B) + V(A \cap B) = V(A) + V(B).$$

Valuations can be thought of as a certain generalization of the notion of measure, and have become a relevant area of study in convex geometry. For instance, volume, surface area, and Euler characteristic are distinguished examples of valuations (in the appropriate classes of sets). Historically, valuations were an essential tool in M. Dehn's solution to Hilbert's third problem, asking whether an elementary definition for volume of polytopes was possible.

The celebrated theorem of H. Hadwiger characterizes continuous rotation and translation invariant valuations on convex bodies as linear combinations of the quermassintegrals [14]. More recently, S. Alesker provided the characterization of those valuations which are only rotation invariant [1], as well as those which are only translation invariant [2]. We refer to [1,2,20–22] for a broad vision on the role of valuations in convex geometry. Recent developments in valuation theory and its connections with other areas of mathematics can also be found in [3].

Valuations on convex bodies are intimately related to the classical Brunn–Minkowski theory. In [23], E. Lutwak introduced and developed a dual version of Brunn–Minkowski theory: in this context, convex bodies, Minkowski addition and Hausdorff metric are replaced by star bodies, radial addition and radial metric, respectively. These have played an important role in the solution of the well-known Busemann–Petty problem [12,13,28], and have become a fundamental area of research [16,24,25]. D. A. Klain initiated in [18], [19] the study of rotationally invariant valuations on a specific class of star-shaped sets, namely those whose radial functions are n-th power integrable.

In this work we characterize radial continuous valuations on $\mathcal{S}_0^n$, the star bodies of $\mathbb{R}^n$ (i.e. star sets with continuous radial function), in terms of an integral representation.

Let $\mathbb{R}_+ := [0, +\infty)$. Our main result is

Theorem 1.1. *$V : \mathcal{S}_0^n \longrightarrow \mathbb{R}$ is a radial continuous valuation if and only if there exist a finite Borel measure μ on S^{n-1} and a function $K : \mathbb{R}_+ \times S^{n-1} \to \mathbb{R}$ such that*

(a) *K satisfies the strong Carathéodory condition (i.e., for each $s \in \mathbb{R}_+$ the function $K(s, \cdot)$ is Borel measurable, and for μ-almost every $t \in S^{n-1}$ the function $K(\cdot, t)$ is continuous),*
(b) *for every $\lambda > 0$ there is $G_\lambda \in L^1(\mu)$ such that $K(s,t) \le G_\lambda(t)$ for $s < \lambda$ and μ-almost every $t \in S^{n-1}$,*