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Livsic-type determinantal representations and hyperbolicity ☆

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ABSTRACT

Hyperbolic homogeneous polynomials with real coefficients, i.e., hyperbolic real projective hypersurfaces, and their determinantal representations, play a key role in the emerging field of convex algebraic geometry. In this paper we consider a natural notion of hyperbolicity for a real subvariety $X \subset \mathbb{P}^d$ of an arbitrary codimension ℓ with respect to a real $\ell - 1$-dimensional linear subspace $V \subset \mathbb{P}^d$ and study its basic properties. We also consider a class of determinantal representations that we call Livsic-type and a nice subclass of these that we call very reasonable. Much like in the case of hypersurfaces ($\ell = 1$), the existence of a definite Hermitian very reasonable Livsic-type determinantal representation implies hyperbolicity. We show that every curve admits a very reasonable Livsic-type determinantal representation. Our basic tools are Cauchy kernels for line bundles and the notion of the Bezoutian for two meromorphic functions on a compact Riemann surface that we introduce. We then proceed to show that every real curve in $\mathbb{P}^d$ hyperbolic with respect to some real $d - 2$-dimensional linear subspace admits a definite

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Hermitian, or even definite real symmetric, very reasonable Livsic-type determinantal representation.

1. Introduction

The study of hyperbolic polynomials originated from the theory of partial differential equations. A linear partial differential equation with constant coefficients is called hyperbolic if there exists $a \in \mathbb{P}^d(\mathbb{R})$, such that the symbol p, considered as a homogeneous polynomial, satisfies $p(a) \neq 0$ and for every $t \in \mathbb{C}$ we have that $p(a+tx) = 0$ only if $t \in \mathbb{R}$ for every $x \in \mathbb{P}^d(\mathbb{R})$. This led Gärding [19,20] and Lax [32] to consider such polynomials and the hypersurfaces $X(\mathbb{R}) = \{x \in \mathbb{P}^d(\mathbb{R})\colon p(x) = 0\}$ they define. In particular, Gärding proved in [20] that if p is hyperbolic with respect to a as above then the connected component C of a in $\mathbb{P}^d(\mathbb{R}) \setminus X(\mathbb{R})$ (with the classical topology) is convex, i.e., the cone over C in $\mathbb{R}^{d+1}$ is a disjoint union of a convex cone and its negative. Furthermore, p is hyperbolic with respect to any b in C (in the case when X is irreducible or $X(\mathbb{R})$ is smooth, C simply consists of all $b \in \mathbb{P}^d(\mathbb{R})$ such that p is hyperbolic with respect to b). More precisely, the cone over the set C in $\mathbb{R}^{d+1}$ has two connected components, each one a convex cone. During the last two decades these hyperbolicity cones came to play an important role in optimization and related fields [8,24,39]. Among other applications, hyperbolic polynomials played a key role in the recent proof by Marcus, Spielman and Srivastava of the Kadison–Singer conjecture in operator algebras [35].

A simple way to manufacture hyperbolic polynomials is to consider Hermitian matrices $A_0, \dots A_d$ such that $A_0 > 0$, and set $p(x_0, \dots, x_d) = \det\left(\sum_{j=0}^d x_j A_j\right)$. Then since $A_0 > 0$, we see easily (using the fact the eigenvalues of a Hermitian matrix are real) that p is hyperbolic with respect to $(1 : 0 : \dots : 0)$. Furthermore, the connected component of $(1, 0, \dots, 0)$ in $\{x \in \mathbb{R}^{d+1}\colon p(x) \neq 0\}$ is given by the linear matrix inequality $\sum_{j=0}^d x_j A_j > 0$, i.e., the hyperbolicity cone is a spectrahedral cone [38] which is the feasible set of a semidefinite program, see [36,37,43] as well as the recent survey volume [10]. In this case we say that p admits a definite Hermitian determinantal representation.

Using the correspondence between determinantal representations and kernel line bundles [44] that goes in its essence back to Dixon [15], and a detailed analysis of the real structure of the corresponding Jacobian variety, it was shown by the second author in [45] that for a smooth real hyperbolic curve in $\mathbb{P}^2$, definite determinantal representations are parametrized by points on a certain distinguished real torus in the Jacobian. In particular, every smooth real hyperbolic curve in $\mathbb{P}^2$ admits a definite determinantal representation, a fact established previously by Dubrovin [16]. A technique using the Cauchy kernels for vector bundles was developed in [7] (following [6]) to provide a construction of determinantal representations for any plane algebraic curve. This technique was later used by Helton and the second author in [28] to prove that every real