Advances in Mathematics 329 (2018) 523–540

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Level structures on Abelian varieties, Kodaira dimensions, and Lang's conjecture [☆]

Dan Abramovich [a], Anthony Várilly-Alvarado [b,*]

[a] *Department of Mathematics, Box 1917, Brown University, Providence, RI, 02912, USA*

[b] *Department of Mathematics MS 136, Rice University, 6100 S. Main St., Houston, TX 77005, USA*

ARTICLE INFO

Article history:
Received 19 November 2016
Received in revised form 15 December 2017
Accepted 19 December 2017
Available online 26 February 2018
Communicated by Ravi Vakil

MSC:
primary 14K10, 14K15
secondary 11G18

Keywords:
Abelian varieties
Moduli spaces
Birational geometry
Rational points

ABSTRACT

Assuming Lang's conjecture, we prove that for a prime p, number field K, and positive integer g, there is an integer r such that no principally polarized abelian variety A/K has full level-p^r structure. To this end, we use a result of Zuo to prove that for each closed subvariety X in the moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g, there exists a level m_X such that the irreducible components of the preimage of X in $\mathcal{A}_g^{[m]}$ are of general type for $m > m_X$.

☆ Research by D.A. partially supported by NSF grant DMS-1500525. Research by A.V.-A. partially supported by NSF CAREER grant DMS-1352291. This paper began as a lunch conversation during the workshop "Explicit methods for modularity of K3 surfaces and other higher weight motives", held at ICERM in October, 2015. We thank the organizers of the workshop and the staff at ICERM for creating the conditions that sparked this project. We also thank Y. Brunebarbe, F. Campana, G. Farkas, B. Hassett, K. Hulek, J. Kollár, R. Lazarsfeld, B. Mazur, M. Popa, C. Schnell, J. Silverman, E. Ullmo, D. Ulmer, J. Voight, D. Zureick-Brown, and the anonymous referees, who gave insightful comments and suggested numerous references.

* Corresponding author.
E-mail addresses: abrmovic@math.brown.edu (D. Abramovich), av15@rice.edu (A. Várilly-Alvarado).

https://doi.org/10.1016/j.aim.2017.12.023

Contents

1. Introduction

1.1. Main result: arithmetic

Yuri Manin proved in [26] that, given a number field K and a prime p, the order of p-primary torsion points across *all* elliptic curves over K is bounded. Our main arithmetic result is an analogous statement for higher dimensional abelian varieties, conditional on Lang's conjecture ([23, Conjecture 5.7], see Conjecture 1.15 below). Instead of p-primary torsion, we treat the more tractable case of full level structures: a *full level-m structure* on an abelian variety A of dimension g is an isomorphism of group schemes on the m-torsion subgroup

$$A[m] \xrightarrow{\sim} (\mathbb{Z}/m\mathbb{Z})^g \times (\mu_m)^g.$$

We do not require this isomorphism to be compatible with the Weil pairing.

Theorem 1.1 *(Uniform power bound). Assume that* Lang*'s conjecture holds. Fix an integer g, a prime number p, and a number field K. Then there is an integer r such that no principally polarized abelian variety A/K of dimension g has full level-p^r structure.*

See §1.4 for known results and variants of Theorem 1.1. The main ingredient in our proof is a powerful result of Zuo [41].[1] The complex function field analogue of our result is shown unconditionally by Hwang and To in [17, Theorem 1.3]. See also Rousseau [35] and Bakker–Tsimerman [3, Theorem A].

Theorem 1.1 is a byproduct of our ongoing pursuit of analogous results for K3 surfaces. That investigation follows on unconditional results of Várilly-Alvarado, with McKinnie, Sawon and Tanimoto in [28] and with Viray in [38]. However, Theorem 1.1 is certainly closer in spirit to numerous unconditional results, of both geometric and arithmetic nature, of Cadoret and Tamagawa, as well as Ellenberg, Hall, and Kowalski; see e.g. [7,8], [12, Theorem 7], and especially Cadoret's conditional result [6].

[1] In an earlier version of this article, we used instead a recent result of Popa and Schnell [33] to get our main argument off the ground. Their result applies to arbitrary families of polarized varieties, not necessarily of Torelli type.