

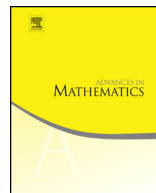


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On divisors of modular forms[☆]

Kathrin Bringmann^{a,*}, Ben Kane^{b,*}, Steffen Löbrich^a,
Ken Ono^c, Larry Rolen^d

^a Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany

^b Department of Mathematics, University of Hong Kong, Pokfulam, Hong Kong

^c Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30022, United States

^d Hamilton Mathematics Institute & School of Mathematics, Trinity College, Dublin 2, Ireland

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ABSTRACT

The *denominator formula* for the Monster Lie algebra is the product expansion for the modular function $J(z) - J(\tau)$ given in terms of the Hecke system of $SL_2(\mathbb{Z})$ -modular functions $j_n(\tau)$. It is prominent in Zagier's seminal paper on traces of singular moduli, and in the Duncan–Frenkel work on Moonshine. The formula is equivalent to the description of the generating function for the $j_n(z)$ as a weight 2 modular form with a pole at z . Although these results rely on the fact that $X_0(1)$ has genus 0, here we obtain a generalization, framed in terms of polar harmonic Maass forms, for all of the $X_0(N)$ modular curves. We use these functions to study divisors of modular forms.

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* Corresponding authors.

E-mail addresses: kbringma@math.uni-koeln.de (K. Bringmann), bkane@hku.hk (B. Kane), s.loeblich@uva.nl (S. Löbrich), ono@mathcs.emory.edu (K. Ono), lrolen@maths.tcd.ie (L. Rolen).

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1. Introduction and statement of results

As usual, let $J(\tau)$ be the $\mathrm{SL}_2(\mathbb{Z})$ Hauptmodul defined by

$$J(\tau) = \sum_{n=-1}^{\infty} c(n) e^{2\pi i n \tau} := \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = e^{-2\pi i \tau} + 196884 e^{2\pi i \tau} + \dots,$$

where $E_k(\tau) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{2\pi i n \tau}$ is the weight $k \in 2\mathbb{N}$ Eisenstein series, $\sigma_\ell(n) := \sum_{d|\ell} d^\ell$, B_k is the k th Bernoulli number, and $\Delta(\tau) := (E_4(\tau)^3 - E_6(\tau)^2)/1728$. By Moonshine (for example, see [14]), $J(\tau)$ is the McKay–Thompson series for the identity (i.e., its coefficients are the graded dimensions of the Monster module $V^\natural$). Moonshine also offers the striking infinite product

$$J(z) - J(\tau) = e^{-2\pi i z} \prod_{m>0, n \in \mathbb{Z}} (1 - e^{2\pi i m z} e^{2\pi i n \tau})^{c(mn)},$$

the *denominator formula* for the Monster Lie algebra. Here we let $\tau, z \in \mathbb{H}$. This formula is equivalent to the following identity of Asai, Kaneko, and Ninomiya (see Theorem 3 of [2])

$$H_z(\tau) := \sum_{n=0}^{\infty} j_n(z) e^{2\pi i n \tau} = \frac{E_4(\tau)^2 E_6(\tau)}{\Delta(\tau)} \frac{1}{J(\tau) - J(z)} = -\frac{1}{2\pi i} \frac{J'(\tau)}{J(\tau) - J(z)}. \quad (1.1)$$

The functions $j_n(\tau)$ form a Hecke system. Namely, if we let $j_0(\tau) := 1$ and $j_1(\tau) := J(\tau)$, then the others are obtained by applying the normalized Hecke operator $T(n)$

$$j_n(\tau) := j_1(\tau) \mid T(n). \quad (1.2)$$

Remark. The functions $H_z(\tau)$ and $j_n(\tau)$ played central roles in Zagier's [20] seminal paper on traces of singular moduli and the Duncan–Frenkel work [13] on the Moonshine Tower. Carnahan [10] has obtained similar denominator formulas for completely replicable modular functions.

If $z \in \mathbb{H}$, then $H_z(\tau)$ is a weight 2 meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a single pole (modulo $\mathrm{SL}_2(\mathbb{Z})$) at the point z . Using these functions, the *divisor modular form* of a normalized weight k meromorphic modular form $f(\tau)$ on $\mathrm{SL}_2(\mathbb{Z})$ was defined in [9] as¹

¹ Note that this summation does not include the cusp $i\infty$.

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