Advances in Mathematics 329 (2018) 541–554

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On divisors of modular forms ☆

Kathrin Bringmann [a,*], Ben Kane [b,*], Steffen Löbrich [a], Ken Ono [c], Larry Rolen [d]

[a] *Mathematical Institute, University of Cologne, Weyertal 86-90, 50931 Cologne, Germany*
[b] *Department of Mathematics, University of Hong Kong, Pokfluam, Hong Kong*
[c] *Department of Mathematics and Computer Science, Emory University, Atlanta, GA 30022, United States*
[d] *Hamilton Mathematics Institute & School of Mathematics, Trinity College, Dublin 2, Ireland*

A R T I C L E   I N F O

Article history:
Received 25 January 2017
Received in revised form 24 January 2018
Accepted 27 January 2018
Available online 26 February 2018
Communicated by George Andrews

In celebration of Don Zagier's 65th birthday

MSC:
11F03
11F37
11F30

A B S T R A C T

The *denominator formula* for the Monster Lie algebra is the product expansion for the modular function $J(z) - J(\tau)$ given in terms of the Hecke system of $\mathrm{SL}_2(\mathbb{Z})$-modular functions $j_n(\tau)$. It is prominent in Zagier's seminal paper on traces of singular moduli, and in the Duncan–Frenkel work on Moonshine. The formula is equivalent to the description of the generating function for the $j_n(z)$ as a weight 2 modular form with a pole at z. Although these results rely on the fact that $X_0(1)$ has genus 0, here we obtain a generalization, framed in terms of polar harmonic Maass forms, for all of the $X_0(N)$ modular curves. We use these functions to study divisors of modular forms.

☆ The first and third author are supported by the Deutsche Forschungsgemeinschaft (DFG) Grant No. BR 4082/3-1. The second author was supported by grant project numbers 27300314, 17302515, and 17316416 of the Research Grants Council. The fourth author thanks the support of the NSF (grant number DMS 1601306) and the Asa Griggs Candler Fund.

* Corresponding authors.
E-mail addresses: kbringma@math.uni-koeln.de (K. Bringmann), bkane@hku.hk (B. Kane), s.loebrich@uva.nl (S. Löbrich), ono@mathcs.emory.edu (K. Ono), lrolen@maths.tcd.ie (L. Rolen).

https://doi.org/10.1016/j.aim.2018.02.001

Keywords:
Divisors of modular forms
Polar harmonic Maass forms
Denominator formula

1. Introduction and statement of results

As usual, let $J(\tau)$ be the $\mathrm{SL}_2(\mathbb{Z})$ Hauptmodul defined by

$$J(\tau) = \sum_{n=-1}^{\infty} c(n)e^{2\pi i n\tau} := \frac{E_4(\tau)^3}{\Delta(\tau)} - 744 = e^{-2\pi i\tau} + 196884e^{2\pi i\tau} + \cdots,$$

where $E_k(\tau) := 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty}\sigma_{k-1}(n)e^{2\pi i n\tau}$ is the weight $k \in 2\mathbb{N}$ Eisenstein series, $\sigma_\ell(n) := \sum_{d|\ell} d^\ell$, B_k is the kth Bernoulli number, and $\Delta(\tau) := (E_4(\tau)^3 - E_6(\tau)^2)/1728$. By Moonshine (for example, see [14]), $J(\tau)$ is the McKay–Thompson series for the identity (i.e., its coefficients are the graded dimensions of the Monster module $V^\natural$). Moonshine also offers the striking infinite product

$$J(z) - J(\tau) = e^{-2\pi i z}\prod_{m>0,\ n\in\mathbb{Z}}\left(1 - e^{2\pi i m z}e^{2\pi i n\tau}\right)^{c(mn)},$$

the *denominator formula* for the Monster Lie algebra. Here we let $\tau, z \in \mathbb{H}$. This formula is equivalent to the following identity of Asai, Kaneko, and Ninomiya (see Theorem 3 of [2])

$$H_z(\tau) := \sum_{n=0}^{\infty} j_n(z)e^{2\pi i n\tau} = \frac{E_4(\tau)^2E_6(\tau)}{\Delta(\tau)}\frac{1}{J(\tau)-J(z)} = -\frac{1}{2\pi i}\frac{J'(\tau)}{J(\tau)-J(z)}. \qquad (1.1)$$

The functions $j_n(\tau)$ form a Hecke system. Namely, if we let $j_0(\tau) := 1$ and $j_1(\tau) := J(\tau)$, then the others are obtained by applying the normalized Hecke operator $T(n)$

$$j_n(\tau) := j_1(\tau) \mid T(n). \qquad (1.2)$$

Remark. The functions $H_z(\tau)$ and $j_n(\tau)$ played central roles in Zagier's [20] seminal paper on traces of singular moduli and the Duncan–Frenkel work [13] on the Moonshine Tower. Carnahan [10] has obtained similar denominator formulas for completely replicable modular functions.

If $z \in \mathbb{H}$, then $H_z(\tau)$ is a weight 2 meromorphic modular form on $\mathrm{SL}_2(\mathbb{Z})$ with a single pole (modulo $\mathrm{SL}_2(\mathbb{Z})$) at the point z. Using these functions, the *divisor modular form* of a normalized weight k meromorphic modular form $f(\tau)$ on $\mathrm{SL}_2(\mathbb{Z})$ was defined in [9] as[1]

[1] Note that this summation does not include the cusp $i\infty$.