Advances in Mathematics 329 (2018) 649–703

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties

E. Arbarello[a], G. Saccà[b,*]

[a] *Dipartimento di Matematica Guido Castelnuovo, Università di Roma Sapienza, Piazzale A. Moro 2, 00185 Roma, Italy*
[b] *Mathematics Department, MIT, Simons Building (Building 2), 77 Massachusetts Ave, Cambridge, MA 02139-4307, United States*

A R T I C L E   I N F O

Article history:
Received 6 October 2016
Accepted 23 January 2018
Available online 23 March 2018
Communicated by Tony Pantev

Keywords:
Moduli spaces
Quiver varieties
K3 surfaces
Symplectic resolution
Wall crossing
Singularities

A B S T R A C T

The aim of this paper is to study the singularities of certain moduli spaces of sheaves on K3 surfaces by means of Nakajima quiver varieties. The singularities in question arise from the choice of a non-generic polarization, with respect to which we consider stability, and admit natural symplectic resolutions corresponding to choices of general polarizations. For sheaves that are pure of dimension one, we show that these moduli spaces are, locally around a singular point, isomorphic to a quiver variety and that, via this isomorphism, the natural symplectic resolutions correspond to variations of GIT quotients of the quiver variety.

Contents

* Corresponding author.
E-mail addresses: ea@mat.uniroma1.it (E. Arbarello), giulia.sacca@stonybrook.edu (G. Saccà).

https://doi.org/10.1016/j.aim.2018.02.003

1. Introduction

A normal variety X is said to have symplectic singularities [3] if its smooth locus X^{sm} carries a holomorphic symplectic form σ having the property that, for any resolution $f : Y \to X$, the pull-back of σ to $f^{-1}(X^{sm})$ extends to a holomorphic form σ_Y on Y. When this is the case, X is called a *symplectic variety*. A resolution $f : Y \to X$ of a symplectic variety is called *symplectic* if, in addition, the holomorphic 2-form σ_Y is non-degenerate. In particular, a symplectic resolution is crepant. Symplectic resolutions are rare: for example, $\mathbb{C}^{2n}/\pm 1$ with the standard symplectic form on the smooth locus is a symplectic singularity, but it admits a symplectic resolution if and only if $n = 1$.

Examples of symplectic varieties and symplectic resolutions come from both representation theory and the theory of moduli spaces of sheaves on K3 or abelian surfaces. Among the symplectic varieties coming from representation theory, we find the nilpotent cone of a complex semisimple Lie algebra and its Springer resolution, the quotients of $\mathbb{C}^2$ by a finite group of symplectic automorphism and their minimal resolutions, and Nakajima quiver varieties. Regarding moduli spaces of sheaves on a K3 surface, their symplectic singularities come from two sources, when the Mukai vector is not primitive, or when the polarization (more generally, the stability condition) is not general. We explain this in Section 2. In [30], Nakajima showed that the Hilbert–Chow morphism, from the Hilbert scheme of points on a holomorphic symplectic surface to the symmetric product of the surface itself, can be described in terms of quiver varieties. This fruitful interaction between quiver varieties and Hilbert schemes of points on surfaces has generated several results, especially on the cohomology and Chow groups of Hilbert schemes. One of the aims of the present article is to generalize Nakajima's description to other moduli spaces and this is the first step in that direction.

Two particular cases of singularities due to a non-primitive Mukai vector were studied by O'Grady [32], [33]. Through this study, he discovered two new examples of irreducible holomorphic symplectic manifolds by exhibiting symplectic resolutions of two singular moduli spaces on a K3 surface and on an abelian surface, respectively. Inspecting O'Grady's construction, Kaledin, Lehn, and Sorger showed, in their inspiring paper [17], that in the remaining cases with non-primitive Mukai vector the corresponding moduli space has no symplectic resolution. Our aim is to continue their investigation, and to study the case when the singularities of a moduli space of sheaves arise from the choice of a non-generic polarization. In certain cases, moving slightly the polarization to a general