Advances in Mathematics 329 (2018) 742–780

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# A solution of the maximality problem for one-parameter dynamical systems

Costel Peligrad

*Department of Mathematical Sciences, University of Cincinnati, 4508 French Hall West, Cincinnati, OH 45221-0025, United States*

ARTICLE INFO

*Article history:*
Received 1 February 2017
Received in revised form 7 February 2018
Accepted 12 February 2018
Available online xxxx
Communicated by Dan Voiculescu

*Keywords:*
One-parameter dynamical system
Connes spectrum
Maximal subalgebra

ABSTRACT

We prove a maximality theorem for one-parameter dynamical systems that include W*-, C*- and multiplier one-parameter dynamical systems. Our main result is new even for one-parameter actions on commutative multiplier algebras including the algebra $C_b(\mathbb{R})$ of bounded continuous functions on $\mathbb{R}$ acted upon by translations. The methods we develop and use in our characterization of maximality include harmonic analysis, topological vector spaces and operator algebra techniques.


## 1. Introduction

The current paper grew out of an effort to find a solution to the general problem of maximality of subalgebras of analytic elements associated to various dynamical systems: W*-dynamical systems, C*-dynamical systems and, as it will follow from the present work, multiplier dynamical systems as defined below (Section 2). The study of maximality of analytic subalgebras associated with C*- or W*-dynamical systems has a history of over six decades. It started with Wermer's maximality theorem [27]. Motivated by some earlier problems about approximation of continuous functions, Wermer showed that if

*E-mail address:* costel.peligrad@uc.edu.

https://doi.org/10.1016/j.aim.2018.02.026

$X = C(\mathbf{T})$ is the C*-algebra of all continuous functions on $\mathbf{T} = \{t \in \mathbb{C} : |t| = 1\}$, then the norm closed subalgebra $A$ of all functions $f \in C(\mathbf{T})$ that have an analytic extension to the unit disk $\mathbf{D} = \{s \in \mathbb{C} : |s| < 1\}$ is a maximal norm-closed subalgebra of $X$. Hoffman and Singer [9] and Simon [23] obtained some generalizations of Wermer's theorem for the case of compact groups with archimedean-linearly ordered Pontryagin duals.

A significant and far reaching generalization of Wermer's maximality theorem was obtained by Forelli [7] using the seminal concepts and results from his paper [6]. In [7], Forelli considered a minimal action of $\mathbb{R}$ on a locally compact Hausdorff space $S$, that is a homomorphism $\tau$ of $\mathbb{R}$ into the group of homeomorphisms of $S$ onto itself such that the mapping $t \to \tau_t(s)$ is continuous for every $s \in S$ and every orbit $\{\tau_t(s) : t \in \mathbb{R}\}, s \in S$ is dense in $S$. If $f \in C_0(S)$ and $t \in \mathbb{R}$, denote $\alpha_t(f) = f \circ \tau_t$. Then $\alpha$ is a homeomorphism of $\mathbb{R}$ into the group of automorphisms of the C*-algebra $X = C_0(S)$ such that the mapping $t \to \alpha_t(f)$ is continuous from $\mathbb{R}$ to $C_0(S)$ endowed with the uniform norm topology for every $f \in C_0(S)$. Forelli proved that the subalgebra $X^\alpha([0,\infty))$ of $X = C_0(S)$ consisting of all functions $f \in C_0(S)$ such that the $\alpha$-spectrum of $f$, $sp_\alpha(f)$, as defined in the next section, is contained in $[0,\infty)$, is a maximal norm closed subalgebra of $X$. An equivalent description of the algebra $X^\alpha([0,\infty))$ for $X = C_0(S)$ and $\alpha$ as defined above is the following: $f \in X^\alpha([0,\infty))$ if and only if the mapping $t \to \alpha_t(f)(s) = (f \circ \tau_t)(s)$ has a bounded analytic extension to the upper half plane for every $s \in S$. In what follows a system $(X, G, \alpha)$ consisting of a C* algebra $X$, a locally compact group $G$ and a homeomorphism $\alpha$ from $G$ into the group $Aut(X)$ of all automorphisms of $X$ such that the mapping $t \to \alpha_t(x)$ is continuous from $G$ to $X$ with the norm topology for every $x \in X$, will be called a C*-dynamical system. When $G = \mathbb{R}$ the system will be called a one-parameter C*-dynamical system. If $X$ does not contain any non trivial norm-closed $\alpha$-invariant ideal, the system will be called $\alpha$-simple. This is the case with Forelli system $(C_0(S), \mathbb{R}, \alpha)$ if $\tau$ is a minimal flow as defined above.

Another direction of study of maximal subalgebras is the following: If $(X, \mathbb{R}, \alpha)$ is a W*-dynamical system consisting of a von Neumann algebra $X$ and an action of $\mathbb{R}$ on $X$ such that the mapping $t \to \alpha_t(x)$ is continuous from $\mathbb{R}$ to $X$ endowed with the $w^*$-topology, for every $x \in X$, when is $X^\alpha([0,\infty))$ a maximal $w^*$-closed subalgebra of $X$? In [21] Sarason noticed that Hoffman and Singer [10] basically proved the first result in this direction: $H^\infty(\mathbf{T})$ is a maximal $w^*$-closed subalgebra of $L^\infty(\mathbf{T})$, where $H^\infty(\mathbf{T}) \subset L^\infty(\mathbf{T})$ is the subalgebra of all $f \in L^\infty(\mathbf{T})$ that have an analytic extension to the unit disk $\mathbf{D}$. Further in [14] Muhly showed that if $(X, \mathbb{R}, \alpha)$ is an ergodic W*-dynamical system with $X$ an abelian von Neumann algebra, then the analytic subalgebra, $X^\alpha([0,\infty))$ as defined below is a maximal w*-closed subalgebra of $X$.

Some extensions of Wermer's result to the more general case of C*- and W*-dynamical systems were obtained in [13] for W*-crossed products and in [18] for C*-crossed products. In [24] Solel has found necessary and sufficient conditions for maximality of subalgebras of analytic elements associated with periodic $\sigma$-finite W*-dynamical systems and, further, in [25] he has considered the maximality for the particular case of an $\alpha$-simple one-parameter $\sigma$-finite W*-dynamical system, i.e. the case when $X$ has no non-trivial