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Triple crystal action in Fock spaces

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ABSTRACT

We make explicit a triple crystal structure on higher level Fock spaces, by investigating at the combinatorial level the actions of two affine quantum groups and of a Heisenberg algebra. To this end, we first determine a new indexation of the basis elements that makes the two quantum group crystals commute. Then, we define a so-called Heisenberg crystal, commuting with the other two. This gives new information about the representation theory of cyclotomic rational Cherednik algebras, relying on some recent results of Shan and Vasserot and of Losev. In particular, we give an explicit labelling of their finite-dimensional simple modules.

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1. Introduction

Since Ariki's proof [1] of the LLT conjecture [22], it is understood that higher level Fock spaces representations of $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$ play, via categorification, a very important role in understanding some classical structures related to complex reflection groups. More precisely, if $\mathcal{F}_{\mathbf{s},e}$ is the level l Fock space representation of $\mathcal{U}_q'(\widehat{\mathfrak{sl}_e})$ with multicharge $\mathbf{s}$ and $V(\mathbf{s})$ the irreducible highest weight submodule of $\mathcal{F}_{\mathbf{s},e}$ of weight $\Lambda_{\mathbf{s}}$ (determined by $\mathbf{s}$), then one can compute the decomposition numbers for the corresponding Ariki–Koike algebra by specialising at $q = 1$ Kashiwara's canonical basis of $V(\mathbf{s})$.

The Fock space itself is no longer irreducible, but one can however define a canonical basis for it, which turns out to give, at $q = 1$, the decomposition numbers of a corresponding q-Schur algebra, as was proved by Varagnolo and Vasserot [35], hence generalising Ariki's result.

The introduction of quiver Hecke algebras by Rouquier [27] and by Khovanov and Lauda [21] has shed some new light about the role of the parameter q. In fact, quiver Hecke algebras are graded, and graded versions of these results (which do not require to specialise q at 1) hold for these structures, see [3].

Moreover, Ariki's categorification theorem also permits to interpret the Kashiwara crystal of $V(\mathbf{s})$ as the branching rule for the associated Ariki–Koike algebra [2]. Shan has proved in [30] that the crystal of the whole Fock space is also categorified by a branching rule, but for another structure, namely a corresponding cyclotomic rational Cherednik algebra.