

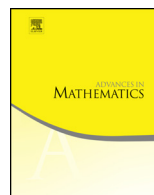


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# Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties

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## ABSTRACT

Motivated by a 1993 conjecture of Stanley and Stembridge, Shareshian and Wachs conjectured that the characteristic map takes the character of the dot action of the symmetric group on the cohomology of a regular semisimple Hessenberg variety to  $\omega X_G(t)$ , where  $X_G(t)$  is the chromatic quasisymmetric function of the incomparability graph  $G$  of the corresponding natural unit interval order, and  $\omega$  is the usual involution on symmetric functions. We prove the Shareshian–Wachs conjecture.

Our proof uses the local invariant cycle theorem of Beilinson–Bernstein–Deligne to obtain a surjection, which we call the local invariant cycle map, from the cohomology of a regular Hessenberg variety of Jordan type  $\lambda$  to a space of local invariant cycles. As  $\lambda$  ranges over all partitions, the local invariant cycles collectively contain all the information about the dot action on a regular semisimple Hessenberg variety. We then prove a result showing that, under suitable hypotheses, the local invariant cycle map is an isomorphism if and only if the special fiber has palindromic cohomology. (This is a general theorem, which is independent of the Hessenberg variety context.) Applying this result to the universal family of Hessenberg varieties, we show that, in our case, the surjections

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are actually isomorphisms, thus reducing the Shareshian–Wachs conjecture to computing the cohomology of a regular Hessenberg variety. But this cohomology has already been described combinatorially by Tymoczko, and, using a new reciprocity theorem for certain quasisymmetric functions, we show that Tymoczko’s description coincides with the combinatorics of the chromatic quasisymmetric function.

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### 1. Introduction

Let  $G$  be the incomparability graph of a unit interval order (also known as an *in-difference graph*), i.e., a finite graph whose vertices are closed unit intervals on the real line, and whose edges join overlapping unit intervals. It is a longstanding conjecture [50] related to various deep conjectures about immanants that if  $G$  is such a graph, then the so-called *chromatic symmetric function*  $X_G$  studied by Stanley [48] is  $e$ -positive, i.e., a nonnegative combination of elementary symmetric functions. (In fact, Stanley and Stembridge conjectured something seemingly more general, but Guay-Paquet [20] has reduced their conjecture to the one stated here.) Early on, Haiman [22] proved that the expansion of  $X_G$  in terms of Schur functions has nonnegative coefficients, and Gasharov [17] showed that these coefficients enumerate certain combinatorial objects known as  *$P$ -tableaux*. It is well known that if  $\chi$  is a character of the symmetric group  $S_n$ , then the image of  $\chi$  under the so-called characteristic map  $\text{ch}$

$$\text{ch } \chi := \frac{1}{n!} \sum_{\sigma \in S_n} \chi(\sigma) p_{\text{cycletype}(\sigma)} \tag{1}$$

(where  $p$  here denotes the power-sum symmetric function) is a nonnegative linear combination of Schur functions, with the coefficients giving the multiplicities of the corresponding irreducible characters of  $S_n$ . One may therefore suspect that  $X_G$  is the image under  $\text{ch}$  of the character of some naturally occurring representation of  $S_n$ , but until recently, there was no candidate, even conjecturally, for such a representation.

Meanwhile, independently and seemingly unrelatedly, De Mari, Procesi, and Shayman [11] inaugurated the study of *Hessenberg varieties*. Let  $\mathbf{m} = (m_1, m_2, \dots, m_{n-1})$  be a weakly increasing sequence of positive integers satisfying  $i \leq m_i \leq n$  for all  $i$ , and let  $s : \mathbb{C}^n \rightarrow \mathbb{C}^n$  be a linear transformation. The (type A) Hessenberg variety  $\mathcal{H}(\mathbf{m}, s)$  is defined by

$$\mathcal{H}(\mathbf{m}, s) := \{ \text{complete flags } F_0 \subseteq F_1 \subseteq \dots \subseteq F_n : sF_i \subseteq F_{m_i} \text{ for } 1 \leq i < n \}. \tag{2}$$

The geometry of a Hessenberg variety depends on the Jordan form of  $s$ . If the Jordan blocks have distinct eigenvalues then we say that  $s$  is *regular*, and, by extension, we also say that  $\mathcal{H}(\mathbf{m}, s)$  is *regular*. Similarly, if  $s$  is diagonalizable then we say that  $\mathcal{H}(\mathbf{m}, s)$  is

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