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Unit interval orders and the dot action on the cohomology of regular semisimple Hessenberg varieties

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ABSTRACT

Motivated by a 1993 conjecture of Stanley and Stembridge, Shareshian and Wachs conjectured that the characteristic map takes the character of the dot action of the symmetric group on the cohomology of a regular semisimple Hessenberg variety to $\omega X_G(t)$, where $X_G(t)$ is the chromatic quasisymmetric function of the incomparability graph G of the corresponding natural unit interval order, and ω is the usual involution on symmetric functions. We prove the Shareshian–Wachs conjecture.

Our proof uses the local invariant cycle theorem of Beilinson–Bernstein–Deligne to obtain a surjection, which we call the local invariant cycle map, from the cohomology of a regular Hessenberg variety of Jordan type λ to a space of local invariant cycles. As λ ranges over all partitions, the local invariant cycles collectively contain all the information about the dot action on a regular semisimple Hessenberg variety. We then prove a result showing that, under suitable hypotheses, the local invariant cycle map is an isomorphism if and only if the special fiber has palindromic cohomology. (This is a general theorem, which is independent of the Hessenberg variety context.) Applying this result to the universal family of Hessenberg varieties, we show that, in our case, the surjections

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are actually isomorphisms, thus reducing the Shareshian–Wachs conjecture to computing the cohomology of a regular Hessenberg variety. But this cohomology has already been described combinatorially by Tymoczko, and, using a new reciprocity theorem for certain quasisymmetric functions, we show that Tymoczko's description coincides with the combinatorics of the chromatic quasisymmetric function.

1. Introduction

Let G be the incomparability graph of a unit interval order (also known as an *indifference graph*), i.e., a finite graph whose vertices are closed unit intervals on the real line, and whose edges join overlapping unit intervals. It is a longstanding conjecture [50] related to various deep conjectures about immanants that if G is such a graph, then the so-called *chromatic symmetric function* X_G studied by Stanley [48] is e-positive, i.e., a nonnegative combination of elementary symmetric functions. (In fact, Stanley and Stembridge conjectured something seemingly more general, but Guay-Paquet [20] has reduced their conjecture to the one stated here.) Early on, Haiman [22] proved that the expansion of X_G in terms of Schur functions has nonnegative coefficients, and Gasharov [17] showed that these coefficients enumerate certain combinatorial objects known as *P-tableaux*. It is well known that if χ is a character of the symmetric group S_n, then the image of χ under the so-called characteristic map ch

$$\operatorname{ch}\chi := \frac{1}{n!}\sum_{\sigma\in S_n}\chi(\sigma)\,p_{\mathrm{cycletype}(\sigma)} \tag{1}$$

(where p here denotes the power-sum symmetric function) is a nonnegative linear combination of Schur functions, with the coefficients giving the multiplicities of the corresponding irreducible characters of S_n. One may therefore suspect that X_G is the image under ch of the character of some naturally occurring representation of S_n, but until recently, there was no candidate, even conjecturally, for such a representation.

Meanwhile, independently and seemingly unrelatedly, De Mari, Procesi, and Shayman [11] inaugurated the study of *Hessenberg varieties*. Let $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$ be a weakly increasing sequence of positive integers satisfying $i \le m_i \le n$ for all i, and let $s : \mathbb{C}^n \to \mathbb{C}^n$ be a linear transformation. The (type A) Hessenberg variety $\mathscr{H}(\mathbf{m}, s)$ is defined by

$$\mathscr{H}(\mathbf{m}, s) := \{\text{complete flags } F_0 \subseteq F_1 \subseteq \cdots \subseteq F_n : sF_i \subseteq F_{m_i} \text{ for } 1 \le i < n\}. \tag{2}$$

The geometry of a Hessenberg variety depends on the Jordan form of s. If the Jordan blocks have distinct eigenvalues then we say that s is *regular*, and, by extension, we also say that $\mathscr{H}(\mathbf{m}, s)$ is *regular*. Similarly, if s is diagonalizable then we say that $\mathscr{H}(\mathbf{m}, s)$ is