Advances in Mathematics 329 (2018) 1137–1155

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

Singularities of mean convex level set flow in general ambient manifolds

Robert Haslhofer [a], Or Hershkovits [b,*]

[a] *Department of Mathematics, University of Toronto, 40 St George Street, Toronto, ON M5S 2E4, Canada*
[b] *Department of Mathematics, Stanford University, 450 Serra Mall, Stanford, CA 94305, USA*

A R T I C L E I N F O

Article history:
Received 6 July 2017
Accepted 23 January 2018
Available online xxxx
Communicated by Camillo De Lellis

Keywords:
Mean curvature flow
Level set flow

A B S T R A C T

We prove two new estimates for the level set flow of mean convex domains in Riemannian manifolds. Our estimates give control – exponential in time – for the infimum of the mean curvature, and the ratio between the norm of the second fundamental form and the mean curvature. In particular, the estimates remove a stumbling block that has been left after the work of White [16,17,20], and Haslhofer–Kleiner [9], and thus allow us to extend the structure theory for mean convex level set flow to general ambient manifolds of arbitrary dimension.

1. Introduction

Let N be a Riemannian manifold. For any *compact* mean convex domain $K_0 \subset N$ with smooth boundary, we consider the level set flow $\{K_t\}_{t\geq 0}$ starting at K_0, i.e. the maximal family of closed sets starting at K_0 that satisfies the avoidance principle when compared with any smooth mean curvature flow [6,7,11]. The level set flow of K_0 coincides with the smooth mean curvature flow of K_0 for as long as the latter is defined, but provides a

* Corresponding author.
E-mail addresses: roberth@math.toronto.edu (R. Haslhofer), orher@stanford.edu (O. Hershkovits).

https://doi.org/10.1016/j.aim.2018.02.008

canonical way to continue the evolution beyond the first singular time. Mean convexity is preserved also beyond the first singular time in the sense that $K_{t_2} \subseteq K_{t_1}$ whenever $t_2 \geq t_1$.

In the last 15 years, Brian White developed a deep regularity and structure theory for mean convex level set flow [16,17,20], and recently the first author and Kleiner gave a new treatment of this theory [9]. Concerning the size of the singular set, white proved that the singular set $\mathcal{S} \subset N^n \times \mathbb{R}$ of any mean convex flow has parabolic Hausdorff dimension at most $n-2$ [16, Thm. 1.1], see also [9, Thm. 1.15]. Concerning the structure of the singular set, the main assertion one wants to prove is that all blowup limits of a mean convex flow are smooth and convex until they become extinct. In particular, one wants to conclude that all tangent flows of a mean convex flow are round shrinking spheres, round shrinking cylinders, or static planes of multiplicity one. While the theorem about the size of the singular set is known in full generality, the structure theorem has been proved up to now only under some additional assumptions [17, Thm. 1], [20, Thm. 3] and [9, Thm. 1.14]. Namely one has to restrict either to blowups at the first singular time, or to low dimensions, or to the case where the ambient manifold is Euclidean space.

As explained in [20, Appendix B], the missing step to extend the structure theorem to general ambient manifolds of arbitrary dimension is to prove that the ratio between the smallest principal curvature λ_1 and the mean curvature H has a finite lower bound on the regular points contained in any compact subset of space–time.

The purpose of this work is to remove this stumbling block. To this end, we prove two new estimates for the level set flow of mean convex domains in Riemannian manifolds.

To state our estimates, we denote by $\partial K_t^{\mathrm{reg}}$ the set of regular boundary points at time t. Our first main estimate gives a lower bound for the mean curvature.

Theorem 1.1 *(Lower bound for H). There exist constants $H_0 = H_0(K_0) > 0$ and $\rho = \rho(K_0) < \infty$ such that*

$$\inf_{\partial K_t^{reg}} H \geq H_0 e^{-\rho t}. \tag{1.2}$$

Our estimate from Theorem 1.1, as well as our second main estimate below, depends exponentially on time. It is clear from simple examples (e.g. flows in hyperbolic space), that this exponential behavior in time is the best one can possibly get.

Our second main estimate, which is the main technical achievement of this paper, controls the ratio between the norm of the second fundamental form and the mean curvature.

Theorem 1.3 *(Upper bound for $|A|/H$). There exist constants $C = C(K_0) < \infty$ and $\rho = \rho(K_0) < \infty$ such that*

$$\sup_{\partial K_t^{reg}} \frac{|A|}{H} \leq C e^{\rho t}. \tag{1.4}$$