Advances in Mathematics 329 (2018) 1231–1284

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

On isometry and isometric embeddability between ultrametric Polish spaces ☆

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A R T I C L E   I N F O

Article history:
Received 7 October 2015
Accepted 27 February 2018
Available online 23 March 2018
Communicated by Slawomir Solecki

MSC:
primary 03E15
secondary 54E50

Keywords:
Borel reducibility
Ultrametric spaces
Polish spaces
Isometry
Isometric embeddability

A B S T R A C T

We study the complexity with respect to Borel reducibility of the relations of isometry and isometric embeddability between ultrametric Polish spaces for which a set D of possible distances is fixed in advance. These are, respectively, an analytic equivalence relation and an analytic quasi-order and we show that their complexity depends only on the order type of D. When D contains a decreasing sequence, isometry is Borel bireducible with countable graph isomorphism and isometric embeddability has maximal complexity among analytic quasi-orders. If D is well-ordered the situation is more complex: for isometry we have an increasing sequence of Borel equivalence relations of length ω_1 which are cofinal among Borel equivalence relations classifiable by countable structures, while for isometric embeddability we have an increasing sequence of analytic quasi-orders of length at least $\omega + 3$.

We then apply our results to solve various open problems in the literature. For instance, we answer a long-standing

☆ Marcone's research was supported by PRIN 2009 Grant "Modelli e Insiemi" and PRIN 2012 Grant "Logica, Modelli e Insiemi". Until September 2014 Motto Ros was a member of the Logic Department of the Albert-Ludwigs-Universität Freiburg, which supported him at early stages of this research. After that, he was supported by the Young Researchers Program "Rita Levi Montalcini" 2012 through the project "New advances in Descriptive Set Theory".

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https://doi.org/10.1016/j.aim.2018.03.001

question of Gao and Kechris by showing that the relation of isometry on locally compact ultrametric Polish spaces is Borel bireducible with countable graph isomorphism.

Contents

1. Introduction

A common problem in mathematics is to classify interesting objects up to some natural notion of equivalence. More precisely, one considers a class of objects X and an equivalence relation E on X, and tries to find a set of complete invariants I for (X, E). To be of any use, such an assignment of invariants should be as simple as possible. In most cases, both X and I carry some intrinsic Borel structures, so that it is natural to ask the assignment to be a Borel measurable map.

A classical example is the problem of classifying separable complete metric spaces, called *Polish metric spaces*, up to isometry. In [13] Gromov showed for instance that one can classify compact Polish metric spaces using (essentially) elements of $\mathbb{R}$ as complete invariants; in modern terminology, we say that the corresponding classification problem is smooth. However, as pointed out by Vershik in [28] the problem of classifying arbitrary Polish metric spaces is "an enormous task", in particular it is far from being smooth. Thus it is natural to ask how complicated is such a classification problem.

A natural tool for studying the complexity of classification problems is the notion of Borel reducibility introduced in [7] and in [14]: we say that a classification problem