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Differentiability and Hölder spectra of a class of self-affine functions

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ABSTRACT

This paper studies a large class of continuous functions $f : [0, 1] \to \mathbb{R}^d$ whose range is the attractor of an iterated function system $\{S_1, \dots, S_m\}$ consisting of similitudes. This class includes such classical examples as Pólya's space-filling curves, the Riesz–Nagy singular functions and Okamoto's functions. The differentiability of f is completely classified in terms of the contraction ratios of the maps $S_1, \dots, S_m$. Generalizing results of Lax (1973) and Okamoto (2006), it is shown that either (i) f is nowhere differentiable; (ii) f is non-differentiable almost everywhere but with uncountably many exceptions; or (iii) f is differentiable almost everywhere but with uncountably many exceptions. The Hausdorff dimension of the exceptional sets in cases (ii) and (iii) above is calculated, and more generally, the complete multifractal spectrum of f is determined.

1. Introduction

In 1973, P. Lax [18] proved a remarkable theorem about the differentiability of Pólya's space-filling curve, which maps a closed interval continuously onto a solid right triangle.

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Unlike the space-filling curves of Peano and Hilbert, which had been known to be nowhere differentiable, Lax found that the differentiability of the Pólya curve depends on the value of the smallest acute angle θ of the triangle. (Roughly speaking, the larger the angle, the less differentiable the function is; see Example 2.3 below.)

More than 30 years later, H. Okamoto [23] introduced a one-parameter family of self-affine functions that includes the Cantor function as well as functions previously studied by Perkins [24] and Katsuura [15]. Okamoto showed that the differentiability of his functions depends on the parameter $a \in (0,1)$ in much the same way as the differentiability of the Pólya curve depends on the angle θ (though it is not clear whether Okamoto was aware of Lax's result). See Example 2.4 below.

While Okamoto's function and the Pólya curve are not directly related, both can be viewed as special cases of a large class of self-affine functions. The aim of this article is to study the differentiability of this class of functions, thereby generalizing the results of Lax and Okamoto, and to determine their finer local regularity behavior in the form of the pointwise Hölder spectrum.

Our class of functions is a subclass of that considered in [4] and may be described as follows. Fix $d \in \mathbb{N}$, an integer $m \geq 2$, and points $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$ with $|\mathbf{a} - \mathbf{b}| = 1$. (Without loss of generality we take $\mathbf{a} = (0,0,\ldots,0)$ and $\mathbf{b} = (1,0,\ldots,0)$.) Fix a vector $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_m) \in \{0,1\}^m$. Let $S_1, \ldots, S_m$ be contractive similitudes in $\mathbb{R}^d$ satisfying the "connectivity conditions"

$$S_1\big((1-\varepsilon_1)\mathbf{a} + \varepsilon_1\mathbf{b}\big) = \mathbf{a}, \tag{1.1}$$

$$S_m\big(\varepsilon_m\mathbf{a} + (1-\varepsilon_m)\mathbf{b}\big) = \mathbf{b}, \tag{1.2}$$

$$S_{i-1}\big(\varepsilon_{i-1}\mathbf{a} + (1-\varepsilon_{i-1})\mathbf{b}\big) = S_i((1-\varepsilon_i)\mathbf{a} + \varepsilon_i\mathbf{b}\big), \qquad i = 2, \ldots, m. \tag{1.3}$$

Put $\lambda_i := \mathrm{Lip}(S_i)$. If $m \geq 3$, we allow one or more of the S_i to be constant, so $\lambda_i = 0$.

Let $c_1, \ldots, c_m$ be positive numbers with $\sum_{i=1}^m c_i = 1$. Put $\sigma_i := \sum_{j=1}^{i-1} c_j + \varepsilon_i c_i$ for $i, \ldots, m$, and define the maps

$$\phi_i(t) := (-1)^{\varepsilon_i} c_i t + \sigma_i, \qquad i = 1, \ldots, m,$$

so ϕ_i maps $[0,1]$ linearly onto a closed interval I_i of length c_i, and the intervals $I_1, \ldots, I_m$ are nonoverlapping with $\bigcup_{i=1}^m I_i = [0,1]$. By a theorem of de Rham [26], there exists a unique continuous function $f : [0,1] \to \mathbb{R}^d$ satisfying the functional equation

$$f(t) = S_i\big(f(\phi_i^{-1}(t))\big), \qquad t \in I_i, \qquad i = 1, \ldots, m. \tag{1.4}$$

Following [4], we shall call $\boldsymbol{\varepsilon}$ the *signature* of f. The image $\Gamma := f([0,1])$ is a connected, self-similar curve in $\mathbb{R}^d$ satisfying $\Gamma = \bigcup_{i=1}^m S_i(\Gamma)$. Note that (1.1)–(1.3) imply that $\sum_{i=1}^m \lambda_i \geq 1$. To avoid degenerate cases, we shall assume throughout that

$$(\lambda_1, \ldots, \lambda_m) \neq (c_1, \ldots, c_m). \tag{1.5}$$