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Monotone, free, and boolean cumulants: A shuffle algebra approach

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ABSTRACT

The theory of cumulants is revisited in the "Rota way", that is, by following a combinatorial Hopf algebra approach. Monotone, free, and boolean cumulants are considered as infinitesimal characters over a particular combinatorial Hopf algebra. The latter is neither commutative nor cocommutative, and has an underlying unshuffle bialgebra structure which gives rise to a shuffle product on its graded dual. The moment-cumulant relations are encoded in terms of shuffle and half-shuffle exponentials. It is then shown how to express concisely monotone, free, and boolean cumulants in terms of each other using the pre-Lie Magnus expansion together with shuffle and half-shuffle logarithms.

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1. Introduction

Since the discovery of free cumulants as the proper way of encoding the notion of free independence in Voiculescu's theory of free probability [22,21], together with their combinatorial description in terms of non-crossing partitions (see e.g. [18], also for further references), the literature on the interplay between the various notions of independence and cumulants has flourished and is still flourishing, as illustrated by recent works such as [1,9,12–14,20].

In [7,8] we proposed a new approach to free cumulants. Consider a non-commutative probability space (A, ϕ) with its unital state $\phi : A \to k$. The central result in [7] is a concise description of the relation between moments and cumulants in free probability in terms of a fixed point equation defined on the graded dual of $T(T(A))$, the double tensor algebra over A equipped with a suitable Hopf algebra structure, which actually is an unshuffle (or codendriform) bialgebra structure. This fixed point equation is solved by using a left half-shuffle exponential. The latter may be considered a natural extension of the notion of time-ordered exponential (also known as Picard or Dyson expansion) familiar in the context of linear differential equations.

In the present work we extend the results in [7] by showing that $T(T(A))$ also encodes in a rather simple manner monotone as well as boolean cumulants. The key to this construction is provided by two other exponential maps (the shuffle and right-half shuffle exponentials), which are naturally defined on the dual of $T(T(A))$. For example, the relation between moments and multivariate monotone cumulants, usually obtained in terms of monotone partitions [12], is recovered by evaluating the shuffle exponential on $T(A)$. The three exponentials allow to compute moments in terms of the corresponding cumulants, and have logarithmic inverses that in return permit to expand cumulants in terms of moments.

From general algebraic arguments, one deduces an intriguing link between the half-shuffle exponentials and the shuffle exponential, given in terms of the pre-Lie Magnus expansion and its compositional inverse. This yields a simple way to express monotone, free, and boolean cumulants concisely in terms of each other. For example, the pre-Lie Magnus expansion permits to express bijectively free cumulants in terms of monotone cumulants. This extends consistently to new algebraic relations among the infinitesimal characters corresponding to monotone, free, and boolean cumulants (Theorem 14). We show, among others, how to recover from this pre-Lie algebra approach the multivariate formulas describing relations between monotone, free, and boolean cumulants in terms of (irreducible) non-crossing partitions presented in reference [1].

The paper is organized as follows. In the next section we show how boolean, free and monotone cumulants can be described as infinitesimal characters on the double tensor algebra defined over a non-commutative probability space. The following section introduces general notions and identities for shuffle (also known as dendriform) algebras and unshuffle (or codendriform) bialgebras. The last section shows how these identi-