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Global well-posedness of critical surface quasigeostrophic equation on the sphere

Diego Alonso-Orán, Antonio Córdoba, Ángel D. Martínez *

Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM – Departamento de Matemáticas (Universidad Autónoma de Madrid), 28049 Madrid, Spain

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ABSTRACT

In this paper we prove global well-posedness of the critical surface quasigeostrophic equation on the two dimensional sphere, building on some earlier work of the authors. The proof relies on an improving of the previously known pointwise inequality for fractional Laplacians as in the work of Constantin and Vicol for the Euclidean setting.

1. Introduction

In this paper we prove global well-posedness of the critical surface quasigeostrophic equation on the two dimensional sphere. The proof relies on an integral representation of the fractional Laplace–Beltrami operator on a general compact manifold and an instantaneous continuity result for weak solutions (cf. Theorems 1.1 and 1.5 stated below). The representation has a benign error term which allows an improvement of the Córdoba–Córdoba inequality [8] in the spirit of Constantin and Vicol (cf. [1,6,7]). Their work

* Corresponding author.
E-mail addresses: diego.alonso@icmat.es (D. Alonso-Orán), antonio.cordoba@uam.es (A. Córdoba), angel.martinez@icmat.es (Á.D. Martínez).

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handles global existence of the critical surface quasigeostrophic equation in $\mathbb{R}^n$, which followed landmark results obtained independently by Kiselev, Nazarov and Volberg [14] and Caffarelli and Vasseur [3]. Underneath our arguments below we exploit the rich group of isometries of the sphere.

1.1. Global well-posedness of the critical surface quasigeostrophic equation on the sphere

The critical surface quasigeostrophic equation on the sphere has been treated in [2] and is given by

$$\begin{cases} \theta_t + u \cdot \nabla_g \theta + \Lambda\theta = 0 \\ u = \nabla_g^{\perp} \Lambda^{-1} \theta \\ \theta(0) = \theta_0 \end{cases} \tag{1.1}$$

One studies the evolution of some class of initial data. In their previous work [2] the authors established an explicit modulus of continuity for weak solutions:

Theorem 1.1. *Given an initial datum $\theta_0 \in L^2(\mathbb{S}^2)$ any weak solution of (1.1) becomes instantaneously continuous with an explicit modulus of continuity, for any time $t > 0$.*

More specifically, if $t \geq t_0 > 0$ then the modulus of continuity is shown to have the form $\omega(\rho) = O\left((\log(1/\rho))^{-\alpha}\right)$ for some fixed $\alpha = \alpha(t_0) > 0$ which degenerates as t_0 tends to zero. The proof exploits Caffarelli and Vasseur's analysis which is based on De Giorgi's techniques on each scale at a time (cf. [2] for details). In the present paper we provide also some regularity results for this equation

Theorem 1.2. *There is global well posedness in $H^s(\mathbb{S}^2)$ for any $s > 3/2$. In fact, any solution with such initial datum becomes smooth instantaneously.*

An immediate consequence is the following result (analogous to that of Nazarov, Kiselev and Volberg [14]).

Theorem 1.3. *Given an initial data $\theta_0 \in C^\infty(\mathbb{S}^2)$, the solution will remain smooth for all times $t > 0$.*

The proof will follow closely the strategy of Constantin and Vicol [6] which is based on the nonlinear maximum principle established with help of the aforementioned explicit integral representation for the fractional Laplace–Beltrami operator. The main difference with their exposition is that we know a priori that the solution is continuous uniformly for times $t \geq t_0 > 0$, invoking Theorem 1.1. This result is of independent interest and allows us to let aside their technical stability result, which might nevertheless be true in $\mathbb{S}^n$. Observe that the modulus of continuity is not uniform for small times. As a