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Integral representation for fractional Laplace–Beltrami operators

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A B S T R A C T

In this paper we provide an integral representation of the fractional Laplace–Beltrami operator for general riemannian manifolds which has several interesting applications. We give two different proofs, in two different scenarios, of essentially the same result. The first deals with compact manifolds with or without boundary, while the second approach treats the case of riemannian manifolds without boundary whose Ricci curvature is uniformly bounded below.

1. Introduction

Fractional laplacians appear in a number of equations of mathematical interest. In the euclidean space and tori explicit expressions are well-known. For example Zygmund's operator, $\Lambda = \sqrt{-\Delta}$, can be expressed in tori as a principal value integral

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$$\Lambda f(x) = c_n P.V. \sum_{\nu \in \mathbb{Z}^n} \int_{\mathbb{T}^n} \frac{f(x) - f(y)}{|x - y - \nu|^{n+1}} dy$$

while in the euclidean space we have

$$\Lambda f(x) = c_n P.V. \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+1}} dy.$$

These expressions combined with elementary algebra provide useful pointwise estimates (see, for instance, [12,11,6]). In other situations, lack of explicit expressions like the one stated above make the analysis much harder to achieve, this typically corresponds to situations involving boundary effects or anisotropy.

In this paper we will provide an integral representation of the fractional Laplace–Beltrami operator on a general compact manifold with a nice error term. As a first example of the power of these explicit formulae we present direct proofs of fractional Sobolev's embeddings on compact manifolds. Furthermore, the explicit kernels make transparent the rôle played by different ad hoc definitions of fractional integration. The authors have applied these representations in their recent work [1,2] to prove global existence of the critical surface quasigeostrophic equation on the two dimensional standard sphere. The present work stems from our original approach [13] which remained unpublished since we found another alternative proof which we judged to be especially elegant. But one of the features of the present approach is that it allows to improve the Córdoba–Córdoba inequality (cf. [12,11,5]) in the spirit of the nonlinear lower bounds due to Constantin and Vicol (cf. [10]), namely, one may achieve pointwise estimates of the form

$$\nabla f(x) \cdot \Lambda^\alpha \nabla f(x) \geq \frac{1}{2} |\nabla f(x)|^2 + \frac{|\nabla f(x)|^{2+\alpha}}{c\|f\|^\alpha_{L^\infty(\mathbb{R}^n)}}.$$

Those turn out to be quite useful concerning the important question of global existence of solutions to the critical surface quasigeostrophic equation in $\mathbb{R}^n$. Their work followed landmark results obtained independently by Kiselev, Nazarov and Volberg [16] and Caffarelli and Vasseur [6]. We believe that the usefulness of this integral representation justify its publication now. It plays an instrumental rôle in the subsequent contributions of the authors to the regularity of weak solutions to the (critical) drift diffusion equations on manifolds, namely

$$\partial_t \theta + u \cdot \nabla_g \theta = -\Lambda \theta$$

where u is a divergence free vector field that depends on θ in some specific manner (e.g. Riesz's transform). Let us highlight the following result for the critical surface quasigeostrophic equation on the sphere.