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Derived localisation of algebras and modules ☆

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ABSTRACT

For any dg algebra A, not necessarily commutative, and a subset S in $H(A)$, the homology of A, we construct its derived localisation $L_S(A)$ together with a map $A \to L_S(A)$, well-defined in the homotopy category of dg algebras, which possesses a universal property, similar to that of the ordinary localisation, but formulated in homotopy invariant terms. Even if A is an ordinary ring, $L_S(A)$ may have non-trivial homology. Unlike the commutative case, the localisation functor does not commute, in general, with homology but instead there is a spectral sequence relating $H(L_S(A))$ and $L_S(H(A))$; this spectral sequence collapses when, e.g. S is an Ore set or when A is a free ring.

We prove that $L_S(A)$ could also be regarded as a Bousfield localisation of A viewed as a left or right dg module over itself. Combined with the results of Dwyer–Kan on simplicial localisation, this leads to a simple and conceptual proof of the topological group completion theorem. Further applications include algebraic K-theory, cyclic and Hochschild homology, strictification of homotopy unital algebras, idempotent ideals,

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the stable homology of various mapping class groups and Kontsevich's graph homology.

Contents

1. Introduction

Localisation of a commutative ring is among the fundamental tools in commutative algebra and algebraic geometry; it has been well-understood and documented for a long time. Let us recall the basic construction.