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Homogeneous bands

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ABSTRACT

A band B is a semigroup such that $e^2 = e$ for all $e \in B$. A countable band is called homogeneous if every isomorphism between finitely generated subbands extends to an automorphism of the band. In this paper, we give a complete classification of all the homogeneous bands. We prove that a homogeneous band belongs to the variety of regular bands, and has a homogeneous structure semilattice.

1. Introduction

A *structure* is a set M together with a collection of finitary operations and relations defined on M. A countable structure M is *homogeneous* if any isomorphism between finitely generated (f.g.) substructures extends to an automorphism of M. Much of the model theoretic interest in homogeneous structures has been due to their strong connections to $\aleph_0$-categoricity and quantifier elimination in the context of *uniformly locally finite* (ULF) structures (see, for example, [9, Theorem 6.4.1]). A structure M is ULF if

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there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every substructure N of M, if N has a generating set of cardinality at most n, then N has cardinality at most $f(n)$. If a ULF structure M has finitely many operations and relations, then the property of homogeneity is equivalent to M being both $\aleph_0$-categorical and having quantifier elimination. It is worth noting that any structure M with no operations (known as *combinatorial*) is ULF, since any subset X is a substructure under the restriction of the relations on M to X.

Consequently, homogeneity of mainly combinatorial structures has been studied by several authors, and complete classifications have been obtained for a number of structures including graphs in [13], partially ordered sets in [19] and (lower) semilattices in [4] and [5]. There has also been significant progress in the classification of homogeneous groups and rings (see, for example, [1], [3] and [18]), and a complete classification of homogeneous finite groups is known ([2], [14]). However, there are a number of fundamental algebraic structures which have yet to be explored from the point of view of homogeneity: in particular semigroups.

In this paper we consider the homogeneity of bands, where a band is a semigroup consisting entirely of idempotents. Our main result is to give a complete description of homogeneous bands. We show that every homogeneous band lies in the variety of *regular* bands and examine how our results fit in with known classifications, in particular showing that the structure semilattice of a homogeneous band is itself homogeneous. We may thus think of classification of homogeneous bands as an extension of the classification of homogeneous semilattices. All structures will be assumed to be countable.

It follows from the work in [15] that bands are ULF, and so we need only to consider isomorphisms between finite subbands. Moreover, as each subsemigroup of a band is a band, we shall write ‘homogeneous bands’ to mean homogeneous as a semigroup, without ambiguity.

In Section 2 we outline the basic theory of bands, and a number of varieties of bands are described. All bands are shown to be built from a semilattice and a collection of simple bands, and the homogeneity of these two building blocks of all bands are completely described. In Section 3 a stronger form of homogeneity is considered and used to obtain a number of results on the homogeneity of regular bands. The structure of a general homogeneous band is also considered, and the results are used in Section 4 to obtain a description of all homogeneous normal bands. In Section 5 we consider the homogeneity of linearly ordered bands, that is, bands with structure semilattice a chain, and these are fully classified. Finally, in Section 6 all other bands are studied and are shown to yield no new homogeneous bands.

2. The theory of bands

Much of the early work on bands was to determine their lattice of varieties: a feat that was independently completed by Biryukov, Fennemore and Gerhard. In addition, Fennemore ([6]) determined all identities on bands, showing that every variety of bands