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Degenerating Hermitian metrics and spectral geometry of the canonical bundle

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ABSTRACT

Let (X,h) be a compact and irreducible Hermitian complex space of complex dimension m. In this paper we are interested in the Dolbeault operator acting on the space of L^2 sections of the canonical bundle of $\mathrm{reg}(X)$, the regular part of X. More precisely let $\overline{\mathfrak{d}}_{m,0} : L^2\Omega^{m,0}(\mathrm{reg}(X),h) \to L^2\Omega^{m,1}(\mathrm{reg}(X),h)$ be an arbitrarily fixed closed extension of $\overline{\partial}_{m,0} : L^2\Omega^{m,0}(\mathrm{reg}(X),h) \to L^2\Omega^{m,1}(\mathrm{reg}(X),h)$ where the domain of the latter operator is $\Omega_c^{m,0}(\mathrm{reg}(X))$. We establish various properties such as closed range of $\overline{\mathfrak{d}}_{m,0}$, compactness of the inclusion $\mathcal{D}(\overline{\mathfrak{d}}_{m,0}) \hookrightarrow L^2\Omega^{m,0}(\mathrm{reg}(X),h)$ where $\mathcal{D}(\overline{\mathfrak{d}}_{m,0})$, the domain of $\overline{\mathfrak{d}}_{m,0}$, is endowed with the corresponding graph norm, and discreteness of the spectrum of the associated Hodge–Kodaira Laplacian $\overline{\mathfrak{d}}^*_{m,0} \circ \overline{\mathfrak{d}}_{m,0}$ with an estimate for the growth of its eigenvalues. Several corollaries such as trace class property for the heat operator associated to $\overline{\mathfrak{d}}^*_{m,0} \circ \overline{\mathfrak{d}}_{m,0}$, with an estimate for its trace, are derived. Finally in the last part we provide several applications to the Hodge–Kodaira Laplacian in the setting of both compact irreducible Hermitian complex spaces with isolated singularities and complex projective surfaces.

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Contents

0. Introduction

Consider a complex projective variety $V \subset \mathbb{CP}^n$. The regular part of V, $\mathrm{reg}(V)$, comes equipped with a natural Kähler metric g, which is the one induced by the Fubini–Study metric of $\mathbb{CP}^n$. In particular, whenever V has a nonempty singular set, we get an incomplete Kähler manifold of finite volume. In the seminal papers [10] and [22], given a singular projective variety V, many questions with a rich interaction of topology and analysis, for instance intersection cohomology, L^2-cohomology and Hodge theory, have been raised for the incomplete Kähler manifold $(\mathrm{reg}(V), g)$. Some of the most important among them are the Cheeger–Goresky–MacPherson's conjecture and the MacPherson's conjecture. The former, which is still open, says that the maximal L^2-de Rham cohomology groups of $(\mathrm{reg}(V), g)$ are isomorphic to the middle perversity intersection cohomology groups of V while the latter, proved in [29], asks whether the L^2-$\overline{\partial}$-cohomology groups in bidegree $(0, q)$ of $(\mathrm{reg}(V), g)$ are isomorphic to the $(0, q)$-Dolbeault cohomology groups of $\tilde{V}$, where $\tilde{V}$ is a resolution of V à la Hironaka. Related to these problems there are many other interesting and deep analytic questions. We can mention for instance the L^2-Stokes theorem which asks whether the maximal and minimal extension of the de Rham differential d are the same, the existence of a L^2-Hodge decomposition for the L^2-de Rham cohomology of $(\mathrm{reg}(V), g)$ in terms of the L^2-$\overline{\partial}$-cohomology of $(\mathrm{reg}(V), g)$, the existence of self-adjoint extensions of the Hodge–de Rham operator $d + d^t$ and the Hodge–Dolbeault operator $\overline{\partial} + \overline{\partial}^t$ with discrete spectrum, the properties of the heat operator associated to some self-adjoint extension of the Laplacian and so on. Moreover we point out that many of these problems admit a natural extension in the more general setting of Hermitian complex spaces. Several papers, during the last thirty years, have been devoted to these questions. Without any goal of completeness we can recall here [18] and [24] which concern the Cheeger–Goresky–MacPherson's conjecture, [9], [15], [28], [27], [29] and [34] devoted to the L^2-$\overline{\partial}$-cohomology, [26] and [33] concerning the $\overline{\partial}$-operator on Hermitian complex spaces, [7], [14], [30] and [35] dealing with the L^2-Hodge decomposition and the L^2-Stokes theorem and finally [8], [20], [23] and [31] devoted to the heat operator.