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Small unions of affine subspaces and skeletons via Baire category ☆

Alan Chang [a], Marianna Csörnyei [a], Kornélia Héra [b], Tamás Keleti [b,*]

[a] Department of Mathematics, The University of Chicago, 5734 S. University Avenue, Chicago, IL 60637, USA
[b] Institute of Mathematics, Eötvös Loránd University, Pázmány Péter sétány 1/c, H-1117 Budapest, Hungary

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ABSTRACT

Our aim is to find the minimal Hausdorff dimension of the union of scaled and/or rotated copies of the k-skeleton of a fixed polytope centered at the points of a given set. For many of these problems, we show that a typical arrangement in the sense of Baire category gives minimal Hausdorff dimension. In particular, this proves a conjecture of R. Thornton.
Our results also show that Nikodym sets are typical among all sets which contain, for every $x \in \mathbb{R}^n$, a punctured hyperplane $H \setminus \{x\}$ through x. With similar methods we also construct a Borel subset of $\mathbb{R}^n$ of Lebesgue measure zero containing a hyperplane at every positive distance from every point.

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* Corresponding author.
E-mail addresses: ac@math.uchicago.edu (A. Chang), csornyei@math.uchicago.edu (M. Csörnyei), herakornelia@gmail.com (K. Héra), tamas.keleti@gmail.com (T. Keleti).

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1. Introduction

E. Stein [19] proved in 1976 that for any $n \geq 3$, if a set $A \subset \mathbb{R}^n$ contains a sphere centered at each point of a set $C \subset \mathbb{R}^n$ of positive Lebesgue measure, then A also has positive Lebesgue measure. It was shown by Mitsis [18] that the same holds if we only assume that C is a Borel subset of $\mathbb{R}^n$ of Hausdorff dimension greater than 1. The analogous results are also true in the case $n = 2$; this was proved independently by Bourgain [2] and Marstrand [15] for circles centered at the points of an arbitrary set $C \subset \mathbb{R}^2$ of positive Lebesgue measure, and by Wolff [21] for $C \subset \mathbb{R}^2$ of Hausdorff dimension greater than 1. In fact, Bourgain proved a stronger result, which extends to other curves with non-zero curvature.

Inspired by these results, the authors in [12] studied what happens if the circles are replaced by axis-parallel squares. They constructed a closed set A of Hausdorff dimension 1 that contains the boundary of an axis-parallel square centered at each point in $\mathbb{R}^2$ (see [12, Theorem 1.1]). Thornton studied in [20] the higher dimensional versions: the problem when $0 \leq k < n$ and $A \subset \mathbb{R}^n$ contains the k-skeleton of an n-dimensional axis-parallel cube centered at every point of a compact set of given dimension d for some fixed $d \in [0, n]$. (Recall that the *k-skeleton* of a polytope is the union of its k-dimensional faces.) He found the smallest possible dimension of such a compact A in the cases when we consider box dimension and packing dimension. He conjectured that the smallest possible Hausdorff dimension of A is $\max(d - 1, k)$, which would be the generalization of [12, Theorem 1.4], which addresses the case $n = 2, k = 0$.

In this paper we prove Thornton's conjecture not only for cubes but for general polytopes of $\mathbb{R}^n$. It turns out that it plays an important role whether 0 is contained in one of the k-dimensional affine subspaces defined by the k-skeleton of the polytope (see Theorem 2.1). This is even more true if instead of just scaling, we also allow rotations. In this case, we ask what the minimal Hausdorff dimension of a set is that contains a scaled and rotated copy of the k-skeleton of a given polytope centered at each point of C. Obviously, it must have dimension at least k if C is nonempty. It turns out that this is sharp: we show that there is a Borel set of dimension k that contains a scaled and rotated copy of the k-skeleton of a polytope centered at each point of $\mathbb{R}^n$, *provided that 0 is not in any of the k-dimensional affine subspaces defined by the k-skeleton.* On the other hand, if 0 belongs to one of these affine subspaces, then the problem becomes much harder (see Remark 3.3).

As mentioned above at the end of the second paragraph, a (very) special case of Theorem 2.1, namely, when $n = 2$ and S consists of the 4 vertices of a square centered at the origin, was already proved in [12]. Our proof of Theorem 2.1 is much simpler than the proof in [12]. In fact, in all our results mentioned above, we will show that, in the sense of Baire category, the minimal dimension is attained by residually many sets. As it often happens, it is much easier to show that some properties hold for residually many sets than to try to construct a set for which they hold. In our case, after proving residuality for