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Classification of nonnegative classical solutions to third-order equations

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ABSTRACT

In this paper, we are concerned with the third-order equations (1.1) and (1.11) with critical or subcritical nonlinearities. By applying the method of moving planes to the third-order PDEs (1.1) and (1.11), we prove that nonnegative classical solutions u to (1.1) and (1.11) are radially symmetric about some point $x_0 \in \mathbb{R}^n$ and derive the explicit forms for u in the critical case. We also prove the non-existence of nontrivial nonnegative classical solutions in the subcritical cases (see Theorem 1.1 and Theorem 1.3).

1. Introduction

In this paper, we mainly consider the nonnegative classical solutions to the following third-order equation with critical or subcritical nonlinearities

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$$\begin{cases}(-\Delta)^{\frac{3}{2}}u(x)=u^p(x), & x\in\mathbb{R}^n,\\ u\in C^{3,\epsilon}_{loc}\cap\mathcal{L}_1(\mathbb{R}^n), \quad u(x)\geq 0, & x\in\mathbb{R}^n,\end{cases} \tag{1.1}$$

where $n\geq 4$, $1\leq p\leq \frac{n+3}{n-3}$, $\epsilon>0$ is arbitrarily small and the fractional Laplacian $(-\Delta)^{\frac{3}{2}}:=(-\Delta)(-\Delta)^{\frac{1}{2}}$. For any $u\in C^{1,\epsilon}_{loc}(\mathbb{R}^n)\cap\mathcal{L}_1(\mathbb{R}^n)$, the nonlocal operator $(-\Delta)^{\frac{1}{2}}$ is defined by (see [9,10,15,27,33])

$$(-\Delta)^{\frac{1}{2}}u(x)=C_n\,P.V.\int_{\mathbb{R}^n}\frac{u(x)-u(y)}{|x-y|^{n+1}}dy:=C_n\lim_{\varepsilon\to 0}\int_{|y-x|\geq\varepsilon}\frac{u(x)-u(y)}{|x-y|^{n+1}}dy, \tag{1.2}$$

where the constant $C_n=\left(\int_{\mathbb{R}^n}\frac{1-\cos(2\pi\zeta_1)}{|\zeta|^{n+1}}d\zeta\right)^{-1}$ and the function space

$$\mathcal{L}_1(\mathbb{R}^n):=\left\{u:\mathbb{R}^n\to\mathbb{R}\,\Big|\int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+1}}dx<\infty\right\}. \tag{1.3}$$

The definition (1.2) of the fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ can also be extended further to distributions in the space $\mathcal{L}_1(\mathbb{R}^n)$ by

$$\langle(-\Delta)^{\frac{1}{2}}u,\phi\rangle=\int_{\mathbb{R}^n}u(x)(-\Delta)^{\frac{1}{2}}\phi(x)dx,\quad \forall\phi\in C_0^\infty(\mathbb{R}^n).$$

The fractional Laplacian $(-\Delta)^{\frac{1}{2}}$ can also be defined equivalently (see [12]) by Caffarelli and Silvestre's extension method (see [1]) for $u\in C^{1,\epsilon}_{loc}(\mathbb{R}^n)\cap\mathcal{L}_1(\mathbb{R}^n)$, that is,

$$(-\Delta)^{\frac{1}{2}}u(x):=-C_n\lim_{y\to 0+}\frac{\partial U(x,y)}{\partial y}=-C_n\lim_{y\to 0+}\int_{\mathbb{R}^n}\frac{|x-\xi|^2-ny^2}{\left(|x-\xi|^2+y^2\right)^{\frac{n+3}{2}}}u(\xi)d\xi, \tag{1.4}$$

where $U(x,y)$ is the harmonic extension of $u(x)$ in $\mathbb{R}^{n+1}_+=\{(x,y)|\,x\in\mathbb{R}^n,\,y\geq 0\}$. Throughout this paper, we define $(-\Delta)^{\frac{3}{2}}u:=(-\Delta)(-\Delta)^{\frac{1}{2}}u$ for $u\in C^{3,\epsilon}_{loc}(\mathbb{R}^n)\cap\mathcal{L}_1(\mathbb{R}^n)$, where $(-\Delta)^{\frac{1}{2}}u$ is defined by definition (1.2) and its equivalent definition (1.4). Due to the nonlocal feature of $(-\Delta)^{\frac{1}{2}}$, we need the assumption $u\in C^{3,\epsilon}_{loc}(\mathbb{R}^n)$ with arbitrarily small $\epsilon>0$ (merely $u\in C^3$ is not enough) to guarantee that $(-\Delta)^{\frac{1}{2}}u\in C^2(\mathbb{R}^n)$ (see [12,27]), and hence u is a classical solution to equation (1.1) in the sense that $(-\Delta)^{\frac{3}{2}}u$ is pointwise well defined and continuous in the whole $\mathbb{R}^n$.

The equation (1.1) is closely related to the following integral equation

$$u(x)=\int_{\mathbb{R}^n}\frac{R_{3,n}}{|x-y|^{n-3}}u^p(y)dy, \tag{1.5}$$

where the Riesz potential's constants $R_{\alpha,n}:=\frac{\Gamma(\frac{n-\alpha}{2})}{\pi^{\frac{n}{2}}2^\alpha\Gamma(\frac{\alpha}{2})}$ for $0<\alpha<n$ (see [28]). When $p=\frac{n+3}{n-3}$, we say the conformally invariant equation (1.1) is $\dot{H}^{\frac{3}{2}}$-critical in the sense