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On a theta lift related to the Shintani lift

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ABSTRACT

We study the Millson theta lift which maps weight $-2k$ to weight $1/2-k$ harmonic weak Maass forms for $k \in \mathbb{Z}, k \geq 0$, and which is closely related to the classical Shintani lift from weight $2k+2$ to weight $k+3/2$ cusp forms. We compute the Fourier expansion of the theta lift and show that it involves twisted traces of CM values and geodesic cycle integrals of the input function. As an application, we recover Zagier's generating series of twisted traces of singular moduli of weight $1/2$ as Millson lifts.

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1. Introduction

A famous result of Zagier [24] states that the twisted traces of singular moduli, i.e., the values of the modular j-invariant at quadratic irrationalities in the upper half-plane, occur as the Fourier coefficients of weakly holomorphic modular forms of weight $1/2$ and $3/2$. Bruinier and Funke [12] showed that the weight $3/2$ generating series of the traces of singular moduli can be obtained as the image of the Kudla–Millson theta lift of $J = j - 744$. Using this approach, new proofs of Zagier's results, including the modularity of generating series of twisted traces of singular moduli, and generalizations to higher weight and level have been studied in several recent works, e.g., [4], [15], [2], [5]. In the

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present work, we study the so-called Millson theta lift which maps weight $-2k$ to weight $1/2-k$ harmonic weak Maass forms, and which (for $k=0$) yields Zagier's generating series of twisted singular moduli of weight $1/2$. We show that the Millson theta lift is related to the classical Shintani lift by the ξ-operator. Further, we compute the Fourier expansion of the Millson lift in terms of traces of CM values and cycle integrals of the input.

To illustrate our results, let us simplify the setup by restricting to modular forms for the full modular group $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. In the body of the paper we also treat forms for arbitrary congruence subgroups by using the theory of vector valued modular forms for the Weil representation of an even lattice of signature $(1,2)$.

We let $z = x+iy \in \mathbb{H}$ and $q = e^{2\pi i z}$. Recall from [11] that a harmonic weak Maass form of weight $k \in \mathbb{Z}$ is a smooth function $F : \mathbb{H} \to \mathbb{C}$ which is invariant under the usual weight k slash operation of Γ, which is annihilated by the weight k hyperbolic Laplace operator Δ_k, and which is mapped to a cusp form of weight $2-k$ under the antilinear differential operator $\xi_k F(z) = 2iy^k \overline{\frac{\partial}{\partial \bar{z}} F(z)}$. The space of such forms is denoted by H_k^+. The subspace of weakly holomorphic modular forms (being holomorphic on $\mathbb{H}$) is denoted by $M_k^!$, and the space of cusp forms is denoted by S_k. Every $F \in H_k^+$ has a Fourier expansion consisting of a holomorphic part F^+ and a non-holomorphic part F^-,

$$F(z) = F^+(z) + F^-(z) = \sum_{n \gg -\infty} a^+(n) q^n + \sum_{n<0} a^-(n) \Gamma(1-k, 4\pi |n| y) q^n,$$

where $\Gamma(s,x) = \int_x^\infty t^{s-1} e^{-t} dt$ is the incomplete Gamma function. Harmonic weak Maass forms of half-integral weight for $\Gamma_0(4)$ are defined analogously.

Let $D \in \mathbb{Z}$ be a discriminant. We let $\mathcal{Q}_D$ be the set of integral binary quadratic forms $Q(x,y) = ax^2+bxy+cy^2$ of discriminant $b^2-4ac = D$. The modular group Γ acts on $\mathcal{Q}_D$, with finitely many classes if $D \neq 0$. For $D<0$ we let $\mathcal{Q}_D^+$ be the subset of positive definite ($a>0$) forms. Further, for $D<0$ the stabilizer $\overline{\Gamma}_Q$ of $Q \in \mathcal{Q}_D$ in $\overline{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ is finite. Let $Q \in \mathcal{Q}_D$. For $D<0$ there is an associated CM point $z_Q \in \mathbb{H}$ which is the solution of $az^2+bz+c=0$, while for $D>0$ the solutions of $a|z|^2+bx+c=0$ define a geodesic c_Q in $\mathbb{H}$, i.e., a vertical line or a semi-circle centered at the real axis, which is equipped with a certain orientation.

Let $\Delta \in \mathbb{Z}$ be a fundamental discriminant (possibly 1). For $k \in \mathbb{Z}_{\geq 0}$ with $(-1)^k \Delta < 0$ the Δ-th Shintani lift of a cusp form $G \in S_{2k+2}$ is (in our normalization) defined by

$$I_\Delta^{\mathrm{Sh}}(G,\tau) = -|\Delta|^{-(k+1)/2} \sum_{\substack{d>0 \\ (-1)^{k+1} d \equiv 0,1(4)}} \left(\sum_{Q \in \mathcal{Q}_{d|\Delta|}/\Gamma} \chi_\Delta(Q) \int_{\Gamma_Q \backslash c_Q} G(z) Q(z,1)^k dz \right) q^d,$$

where χ_Δ is a generalized genus character, which is defined in Section 3.1. It is well known that the Shintani lift of F is a cusp form of weight $3/2+k$ for $\Gamma_0(4)$ which

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