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Monoidal categories associated with strata of flag manifolds

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We construct a monoidal category $\mathscr{C}_{w,v}$ which categorifies the doubly-invariant algebra ${}^{N'(w)}\mathbb{C}[N]^{N(v)}$ associated with Weyl group elements w and v. It gives, after a localization, the coordinate algebra $\mathbb{C}[\mathcal{R}_{w,v}]$ of the open Richardson variety associated with w and v. The category $\mathscr{C}_{w,v}$ is realized as a subcategory of the graded module category of a quiver Hecke algebra R. When $v = \mathrm{id}$, $\mathscr{C}_{w,v}$ is the same as the monoidal category which provides a monoidal categorification of the quantum unipotent coordinate algebra $A_q(\mathfrak{n}(w))_{\mathbb{Z}[q,q^{-1}]}$ given by Kang–Kashiwara–Kim–Oh. We show that the category $\mathscr{C}_{w,v}$ contains special determinantial modules $\mathsf{M}(w_{\le k}\Lambda, v_{\le k}\Lambda)$ for $k = 1, \ldots, \ell(w)$, which commute with each other. When

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the quiver Hecke algebra R is symmetric, we find a formula of the degree of R-matrices between the determinantial modules $\mathsf{M}(w_{\le k}\Lambda, v_{\le k}\Lambda)$. When it is of finite ADE type, we further prove that there is an equivalence of categories between $\mathscr{C}_{w,v}$ and $\mathscr{C}_u$ for $w, u, v \in \mathsf{W}$ with $w = vu$ and $\ell(w) = \ell(v) + \ell(u)$.

Contents

Introduction

A *quantum cluster algebra* is a non-commutative q-deformation of a *cluster algebra* which is a special $\mathbb{Z}$-subalgebra of a rational function field with a set of generators grouped into overlapping subsets, called *clusters*, and a procedure which makes new clusters, called *mutation*. The (quantum) cluster algebras appear naturally in various studies of mathematics including representation theory of Kac–Moody algebras. Let $\mathfrak{g}$ be a Kac–Moody algebra of symmetric type and $U_q(\mathfrak{g})$ its quantum group. We take an element w of the Weyl group W. It was shown in [2] that the quantum unipotent coordinate algebra $A_q(\mathfrak{n}(w))$ associated with $U_q(\mathfrak{g})$ and w has a quantum cluster algebra structure. It turned out that the intersection of the upper global basis and $A_q(\mathfrak{n}(w))$ is a basis of $A_q(\mathfrak{n}(w))$ ([15]). On the other hand, it was proved that there is a cluster algebra structure inside the coordinate ring of a *Richardson variety*. For $w, v \in \mathsf{W}$ with $v \le w$, one can consider the *open Richardson variety* $\mathcal{R}_{w,v}$ which is the intersection of the Schubert cell and the opposite Schubert cell attached to w and v respectively. It was shown in [17] that, when $\mathfrak{g}$ is of finite ADE type, the coordinate algebra $\mathbb{C}[\mathcal{R}_{w,v}]$ contains a subalgebra with the cluster algebra structure coming from a certain subcategory of module category of a preprojective algebra which is determined by w, v. The cluster variables of the initial cluster are given by the irreducible factors of the generalized minors determined by $w_{\le j}$