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Kauffman states, bordered algebras, and a bigraded knot invariant

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ABSTRACT

We define and study a bigraded knot invariant whose Euler characteristic is the Alexander polynomial, closely connected to knot Floer homology. The invariant is the homology of a chain complex whose generators correspond to Kauffman states for a knot diagram. The definition uses decompositions of knot diagrams: to a collection of points on the line, we associate a differential graded algebra; to a partial knot diagram, we associate modules over the algebra. The knot invariant is obtained from these modules by an appropriate tensor product.

1. Introduction

The Alexander polynomial can be given a state sum formulation as a count of certain Kauffman states, each of which contributes a monomial in a formal variable t [6]. In [22], this description was lifted to knot Floer homology [23,25]: knot Floer homology is given

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as the homology of a chain complex whose generators correspond to Kauffman states. The differentials in the complex, though, were not understood explicitly; they were given as counts of pseudo-holomorphic curves. A much larger model for knot Floer homology was described in [14], where the generators correspond to certain states in a grid diagram, and whose differentials count certain embedded rectangles in the torus. The grid diagram can be used to compute invariants for small knots [1,4], but computations are limited by the size of the chain complex (which has $n!$ many generators for a grid diagram of size n).

The aim of this article is to construct and study an invariant of knots, $H^-(K)$, with the following properties.

(H-1) Letting $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$, $H^-(K)$ is a bigraded module over the polynomial algebra $\mathbb{F}[U]$. That is, there is a vector space splitting $H^-(K) \cong \oplus_{d,s} H^-_d(K,s)$, and an endomorphism U of $H^-(K)$ with $U\colon H^-_d(K,s) \to H^-_{d-2}(K,s-1)$.
(H-2) If $\mathcal{D}$ is a diagram for K, with a marked edge, then $H^-(K)$ is obtained as the homology of a chain complex $C^-(\mathcal{D})$ associated to the diagram.
(H-3) The complex $C^-(\mathcal{D})$ is a bigraded chain complex over $\mathbb{F}[U]$, which is freely generated by the Kauffman states of $\mathcal{D}$.
(H-4) The graded Euler characteristic of $H^-(K)$ is related to the symmetrized Alexander polynomial $\Delta_K(t)$ of the knot K, as follows: there is an identification of Laurent series in $\mathbb{Z}[t,t^{-1}]\!]$

$$\sum_{d,s}(-1)^d \dim H^-_d(K,s)t^s = \frac{\Delta_K(t)}{(1-t^{-1})}. \tag{1.1}$$

From its description, this invariant comes equipped with a great deal of algebraic structure, similar to the Khovanov and Khovanov–Rozansky categorifications of the Jones polynomial and its generalizations. The structure also makes computations of the invariant for large examples feasible. In this paper, we also give an algebraic proof of invariance, hence giving a self-contained treatment of this invariant.

Building on the present work, in [19], we generalize the constructions to define an invariant with more algebraic structure. In [17], we relate the constructions here and their generalizations from [19] with pseudo-holomorphic curve counting, to give an identification between these algebraically-defined invariants and a suitable variant of knot Floer homology [17]. See [21] for an expository paper overviewing this material. A generalization of these constructions to links is given in [18].

1.1. Decomposing knot diagrams

The knot invariant $H^-(K)$ is constructed by decomposing a knot projection for K into elementary pieces, and using those pieces to put together a chain complex whose homology is $H^-(K)$.