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The homotopy types of the posets of p-subgroups of a finite group ☆

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We study the homotopy properties of the posets of p-subgroups $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ of a finite group G, viewed as finite topological spaces. We answer a question raised by R.E. Stong in 1984 about the relationship between the contractibility of the finite space $\mathcal{A}_p(G)$ and that of $\mathcal{S}_p(G)$ negatively, and describe the contractibility of $\mathcal{A}_p(G)$ in terms of algebraic properties of the group G.

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1. Introduction

The poset $\mathcal{S}_p(G)$ of non-trivial p-subgroups of a finite group G was introduced by K. Brown in the seventies [8]. Brown observed that the topology of the simplicial complex associated to this poset, which we denote by $\mathcal{K}(\mathcal{S}_p(G))$, is related to the algebraic properties of G, and proved the Homological Sylow Theorem

$$\chi(\mathcal{K}(\mathcal{S}_p(G))) \equiv 1 \mod |G|_p,$$

where $\chi(\mathcal{K}(\mathcal{S}_p(G)))$ denotes the Euler characteristic of the complex and $|G|_p$ is the greatest power of p that divides the order of G.

The study of the topological properties of $\mathcal{S}_p(G)$ was continued by D. Quillen in his seminal paper [16]. Quillen investigated the homotopy properties of $\mathcal{K}(\mathcal{S}_p(G))$ by comparing it with the complex associated to the subposet $\mathcal{A}_p(G)$ of non-trivial elementary abelian p-subgroups of G. He proved that $\mathcal{K}(\mathcal{S}_p(G))$ and $\mathcal{K}(\mathcal{A}_p(G))$ are homotopy equivalent and that these polyhedra are contractible if G has a non-trivial normal p-subgroup. Quillen conjectured that the converse should hold [16, Conjecture 2.9]: if $\mathcal{K}(\mathcal{S}_p(G))$ is contractible then $\mathcal{O}_p(G) \neq 1$. Here $\mathcal{O}_p(G)$ denotes the maximal normal p-subgroup of G. The conjecture remains unproven but there have been remarkable progresses. In the nineties M. Aschbacher and S. Smith obtained the most significant partial confirmation of Quillen's conjecture so far [3].

The works of Brown and Quillen on the topology of the p-subgroup complexes have been pursued by many mathematicians, who related the topological properties of the complexes and the combinatorics of the posets with the algebraic properties of the group (see [2,1,7,9,11,13,14,17,22,23]). For example, in [1,13,14] the authors investigated the fundamental group of these complexes and in [11] T. Hawkes and I.M. Isaacs proved that if G is p-solvable and has an abelian Sylow p-subgroup, then $\chi(\mathcal{K}(\mathcal{S}_p(G))) = 1$ if and only if $\mathcal{O}_p(G) \neq 1$. We refer the reader to S. Smith's book [18] for more details on subgroup complexes and the development of these results along the last decades.

In all the articles that we mentioned above, the authors handled the posets $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ topologically by means of their classifying spaces (or order complexes) $\mathcal{K}(\mathcal{S}_p(G))$ and $\mathcal{K}(\mathcal{A}_p(G))$. In 1984 R.E. Stong adopted an alternative point of view: he treated $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ as finite topological spaces [21]. Any finite poset has an intrinsic topology and in [21] Stong used results on the homotopy theory of finite spaces that he obtained previously in [20] and results of McCord [15], in order to relate the (intrinsic) topology of the posets $\mathcal{S}_p(G)$ and $\mathcal{A}_p(G)$ with the algebraic properties of the group. At that time it was already known that for any finite poset X (viewed as a finite space) there exists a natural weak equivalence $\mu_X : \mathcal{K}(X) \to X$. In particular, the posets $\mathcal{A}_p(G)$ and $\mathcal{S}_p(G)$ are weak equivalent (viewed as finite spaces) since their order complexes are homotopy equivalent. But the notion of homotopy equivalence and contractibility in the context of finite topological spaces is strictly stronger than those in the context of polyhedra. This