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Trilinear forms with Kloosterman fractions

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We give new bounds for $\sum_{a,m,n} \alpha_m \beta_n \nu_a \, \mathrm{e}\left(\frac{a\overline{m}}{n}\right)$ where α_m, β_n and ν_a are arbitrary coefficients, improving upon a result of Duke, Friedlander and Iwaniec [7]. We also apply these bounds to problems on representations by determinant equations and on the equidistribution of solutions to linear equations.

1. Introduction

For a, b, c positive integers, one defines the classical Kloosterman sum as

$$S(a,b;c) := \sum_{x \pmod{c}}^{*} \mathrm{e}\left(\frac{a\overline{x}+bx}{c}\right)$$

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where, as usual, $\overline{x}$ denotes the multiplicative inverse of x modulo the denominator c, $\sum^*$ denotes a sum over the reduced residues modulo c, and $\mathrm{e}(x) := e^{2\pi i x}$.

Several important results in number theory have been obtained by using bounds for single Kloosterman sums such as Weil's bound, $S(a,b,c) \ll (a,b,c)^{\frac{1}{2}} c^{\frac{1}{2}+\varepsilon}$, or, more recently, for averages of Kloosterman sums, in particular the bounds of Deshouillers and Iwaniec [5].

The results of [5] are particularly efficient when considering averages of $S(a,b,c)$ with weights $f(a,b,c)$ that are smooth or have at least some special structure. For many applications, however, one would like to have non-trivial bounds in the case of arbitrary weights and such bounds would be useful also when the coefficients have arithmetic/geometric nature, but have conductor which is too large to be able to employ the extra information.

In the beautiful paper [7], Duke, Friedlander and Iwaniec addressed this problem, obtaining a non-trivial bound for the following "bilinear form with Kloosterman fractions":

$$\mathcal{B}_a(M,N) := \sum_{\substack{m\in\mathcal{M}, n\in\mathcal{N},\\ (m,n)=1}}\sum \alpha_m \beta_n \,\mathrm{e}\left(\frac{a\overline{m}}{n}\right),$$

where α_m, β_n are arbitrary coefficients supported on $\mathcal{M} := [M/2, M]$ and $\mathcal{N} := [N/2, N]$ respectively, and $a \neq 0$. The main result of [7] is the bound

$$\mathcal{B}_a(M,N) \ll \|\alpha\|\|\beta\|(|a| + MN)^{\frac{3}{8}}(M+N)^{\frac{11}{48}+\varepsilon}, \tag{1.1}$$

where $\|\cdot\|$ denotes the L^2-norm. Notice that, in the important case $M \approx N$, $a \ll MN$, the bound in (1.1) saves roughly a power of $N^{\frac{1}{48}}$ over the trivial bound $\mathcal{B}_a(M,N) \ll \|\alpha\|\|\beta\|(MN)^{\frac{1}{2}}$. In this paper, we refine the arguments of Duke, Friedlander and Iwaniec and improve upon their bound, obtaining

$$\mathcal{B}_a(M,N) \ll \|\alpha\|\|\beta\|(|a| + MN)^{\frac{1}{2}}(MN)^{-\frac{3}{20}+\varepsilon}(M+N)^{\frac{1}{4}}.$$

We get a saving of $N^{\frac{1}{20}}$ when $M \approx N$, $a \ll MN$. More generally, we consider the case when an extra average over a is introduced, as this often appears in applications. We then provide a new bound for sums of the form

$$\mathcal{B}(M,N,A) := \sum_{\substack{a\in\mathcal{A}, m\in\mathcal{M}, n\in\mathcal{N},\\ (m,n)=1}}\sum\sum \alpha_m \beta_n \nu_a \,\mathrm{e}\left(\vartheta\frac{a\overline{m}}{n}\right),$$

where ν_a are arbitrary coefficients supported on $\mathcal{A} := [A/2, A]$ and $\vartheta \neq 0$.