ARTICLE IN PRESS YAIMA:5892

Advances in Mathematics ••• (••••) •••–•••

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

The first fundamental theorem of invariant theory for the orthosymplectic super group

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ARTICLE INFO

Article history:
Available online xxxx

To David Kazhdan, with admiration

MSC:
primary 15A72
secondary 16A22, 14M30

Keywords:
Super algebraic geometry
Orthosymplectic group
Periplectic group
Super Pfaffian
Tensor invariant

ABSTRACT

We give a new proof, inspired by an argument of Atiyah, Bott and Patodi, of the first fundamental theorem of invariant theory for the orthosymplectic super group. We treat in a similar way the case of the periplectic super group. Lastly, the same method is used to explain the fact that Sergeev's super Pfaffian, an invariant for the special orthosymplectic super group, is polynomial.

Contents

1. Introduction . . . 2
2. Tools from algebraic geometry . . . 3
3. Reduction of the FFT for OSp to the FFT for GL . . . 10
4. Periplectic analog . . . 15
5. Super Pfaffian . . . 18
References . . . 21

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http://dx.doi.org/10.1016/j.aim.2017.06.009

1. Introduction

1.1. Let V be a finite dimensional complex vector space, with a non-degenerate symmetric bilinear form B. The first fundamental theorem (FFT) for the orthogonal group $\mathrm{O}(V)$ gives generators for the linear space of $\mathrm{O}(V)$-invariant multilinear forms on $V \times V \times \cdots \times V$ (N factors). The generators are obtained from partitions $\mathcal{P}$ of $\{1, \ldots, N\}$ into subsets with two elements: to each $\mathcal{P}$ corresponds the multilinear form

$$v_1, \ldots, v_N \mapsto \prod_{\{i_1, i_2\} \in \mathcal{P}} B(v_{i_1}, v_{i_2}). \tag{1.1.1}$$

This theorem is proved in Weyl [11] using the Capelli identity. In appendix 1 to [1], Atiyah, Bott and Patodi suggested a more geometric approach, reducing the FFT for $\mathrm{O}(V)$ to the FFT for $\mathrm{GL}(V)$. The latter gives generators for the linear space of $\mathrm{GL}(V)$-invariant multilinear forms on $V \times \cdots \times V \times V^\vee \times \cdots \times V^\vee$: the generators arise from pairings between the factors V and $V^\vee$, by a formula similar to (1.1.1). The FFT for $\mathrm{GL}(V)$ is equivalent to the Schur–Weyl duality between $\mathrm{GL}(V)$ and the symmetric group S_M, both acting on $V^{\otimes M}$.

1.2. We show that this idea can be made to work in the "super world", where one systematically considers $\mathbb{Z}/2$-graded objects, and where for super vector spaces (that is, $\mathbb{Z}/2$-graded vector spaces) V and W, the isomorphism $V \otimes W \to W \otimes V$ is defined to be

$$v \otimes w \mapsto (-1)^{|v||w|} w \otimes v \tag{1.2.1}$$

for v and w homogeneous of degrees $|v|$ and $|w|$ (sign rule). For an explanation of how concepts of algebraic or differential geometry extend to the super world, we refer to Leites [6], Manin [7], or Bernstein, Deligne and Morgan [2], as well as to §2 below.

In the super world, V is taken to be a super vector space, and B to be a non-degenerate bilinear form, symmetric in the sense of being invariant by (1.2.1). The FFT describes the multilinear forms on $V \times V \times \cdots \times V$, invariant under the algebraic super group $\mathrm{O}(V)$. They are derived from B by (1.1.1). This was announced by Sergeev in [10], and we obtain a new proof of this FFT in §3. Another approach, also inspired by [1], using as ring of coefficients the infinite dimensional Grassmann algebra, may be found in [3]. In [4], this reduction to the case of $\mathrm{GL}(V)$ leads to a new second fundamental theorem (SFT) which describes all relations among the generators.

The form B can be viewed as a morphism

$$V \otimes V \to \mathbb{C}$$

invariant by (1.2.1). "Morphism" implies "compatible with the $\mathbb{Z}/2$-grading". As