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A trichotomy theorem for transformation groups of locally symmetric manifolds and topological rigidity

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To our friend, David Kazhdan, with deep admiration and affection

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ABSTRACT

Let M be a locally symmetric irreducible closed manifold of dimension ≥ 3. A result of Borel [6] combined with Mostow rigidity imply that there exists a finite group $G = G(M)$ such that any finite subgroup of $\mathrm{Homeo}^+(M)$ is isomorphic to a subgroup of G. Borel [6] asked if there exist M's with $G(M)$ trivial and if the number of conjugacy classes of finite subgroups of $\mathrm{Homeo}^+(M)$ is finite. We answer both questions:

(1) For every finite group G there exist M's with $G(M) = G$, and
(2) the number of maximal subgroups of $\mathrm{Homeo}^+(M)$ can be either one, countably many or continuum and we determine (at least for $\dim M \neq 4$) when each case occurs.

Our detailed analysis of (2) also gives a complete characterization of the topological local rigidity and topological strong rigidity (for $\dim M \neq 4$) of proper discontinuous actions of uniform lattices in semisimple Lie groups on the associated symmetric spaces.

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1. Introduction

A positive dimensional, oriented, closed manifold M has a very large group of automorphisms (i.e., orientation preserving self-homeomorphisms). In fact this group $\mathrm{Homeo}^+(M)$ is infinite dimensional. But its finite subgroups are quite restricted. In 1969, Borel showed (in a classic paper [6] but which appeared only in 1983 in his collected works) that if M is a $K(\pi, 1)$-manifold with fundamental group $\Gamma = \pi_1(M)$, whose center is trivial, then every finite transformation group G in $\mathrm{Homeo}^+(M)$ is mapped injectively into the outer automorphism group $\mathrm{Out}(\Gamma)$ by the natural map (or more precisely into the subgroup $\mathrm{Out}^+(\Gamma)$ – see §2 – which has an index at most 2 in $\mathrm{Out}(\Gamma)$).

Let now M be a locally symmetric manifold of the form $\Gamma\backslash H/K$ when H is a connected non-compact semisimple group with trivial center and with no compact factor and Γ a torsion free uniform irreducible lattice in H. In the situation in which strong rigidity (in the sense of Mostow [39]) holds (i.e., if H is not locally isomorphic to $SL_2(\mathbb{R})$), $\mathrm{Out}(\Gamma)$ is a finite group, $G \leq \mathrm{Out}^+(\Gamma)$; in fact, $\mathrm{Out}^+(\Gamma) \subseteq N_H(\Gamma)/\Gamma$ and it acts on M as the group of (orientation preserving) self-isometries $\mathrm{Isom}^+(M)$ of the Riemannian manifold M. It follows now from Borel's theorem that every finite subgroup of $\mathrm{Homeo}^+(M)$ is isomorphic to a subgroup of one finite group, $G(M) = \mathrm{Isom}^+(M)$.

Borel ends his paper by remarking: "The author does not know whether the finite subgroups of $\mathrm{Homeo}^+(M)$ form finitely many conjugacy classes, nor whether one can find a Γ with no outer automorphism."

The goal of the current paper is to answer these two questions. For an efficient formulation of our results, let us make the following definition(s):

Definition 1.1. Let G be a finite group. An oriented manifold M will be called G-dominated (resp., G-weakly-dominated) if there is a faithful action of G on M, so that G can be identified with a subgroup of $\mathrm{Homeo}^+(M)$ and if F is any finite subgroup of $\mathrm{Homeo}^+(M)$, then F is conjugate (resp., isomorphic) to a subgroup of G.

Note that Borel's Theorem combined with strong rigidity implies that unless H is locally isomorphic to $SL_2(\mathbb{R})$, M as above is always at least $\mathrm{Isom}^+(M)$-weakly-dominated.

We now assert (to be proved in section 2)

Theorem 1.2. *For every finite group G and every $3 \leq n \in \mathbb{N}$, there exist infinitely many oriented closed hyperbolic manifolds $M = M^n(G)$ of dimension n, with $G \simeq \mathrm{Isom}^+(M)$ and when $n \neq 4$ these $M^n(G)$ are also G-dominated.*

The very special case $G = \{e\}$ answers Borel's second question (where one can also deduce it from [2]). It also answers the question of Schultz [41,42], attributed there to D. Burghelea, who asked whether there exist asymmetric closed manifolds with degree one maps onto hyperbolic manifolds. Our examples are even hyperbolic themselves.

The situation for dimension 2 is very different: