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Koszul duality and the PBW theorem in symmetric tensor categories in positive characteristic

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To David Kazhdan on his 70th birthday with admiration

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ABSTRACT

We generalize the theory of Koszul complexes and Koszul algebras to symmetric tensor categories. In characteristic zero the generalization is routine, while in characteristic p there is a subtlety – the symmetric algebra of an object is not always Koszul (i.e., its Koszul complex is not always exact). Namely, this happens in the Verlinde category Ver_p in any characteristic $p \geq 5$. We call an object Koszul if its symmetric algebra is Koszul, and show that the only Koszul objects of Ver_p are usual supervector spaces, i.e., a non-invertible simple object L_m $(2 \leq m \leq p-2)$ is not Koszul. We show, however, that the symmetric algebra SL_m is almost Koszul in the sense of Brenner, Butler and King (namely, $(p-m, m)$-Koszul), and compute the corresponding internal Yoneda algebra (i.e., the internal Ext-algebra from the trivial module to itself).
We then proceed to discuss the PBW theorem for operadic Lie algebras (i.e., algebras over the operad **Lie**). This theorem is well known to fail for vector spaces in characteristic 2 (as one needs to require that $[x, x] = 0$), and for supervector spaces in characteristic 3 (as one needs to require that $[[x, x], x] = 0$ for odd x), but it holds in these categories in any characteristic $p \geq 5$; there is a well known proof based on Koszul duality. However, we show that in the category Ver_p, because of failure of Koszul duality, the PBW theorem can fail in any characteristic $p \geq 5$. Namely, one needs to impose the p-Jacobi identity, a certain generalization to characteristic p of the identities $[x, x] = 0$ and $[[x, x], x] = 0$. On the other hand, our main result is that once the p-Jacobi identity is imposed, the PBW theorem holds. This shows that the correct definition of

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a Lie algebra in Ver_p is an algebra over **Lie** which satisfies the p-Jacobi identity. This also applies to any symmetric tensor category that admits a symmetric tensor functor to Ver_p (e.g., a symmetric fusion category, see [19], Theorem 1.5). Finally, we prove the PBW theorem for Lie algebras in any quasi-semisimple symmetric tensor category.

1. Introduction

The goal of this paper is to extend the theory of Koszul duality and Koszul algebras to symmetric tensor categories, and to use it to prove the PBW theorem for Lie algebras in such categories.

We start by defining the Koszul complex $K^\bullet(V)$ of an object V of a symmetric tensor category $\mathcal{C}$ over a field $\mathbf{k}$ (following [7], Subsection 3.4). In the classical setting (of vector and supervector spaces), this complex is exact, which is a basic fact in commutative algebra. In particular, it is exact in the category of Schur functors, and hence in any symmetric tensor category if $\mathrm{char}(\mathbf{k}) = 0$.

In characteristic p, however, the story is more complicated, and the Koszul complex may fail to be exact. Thus we call an object V Koszul if its Koszul complex is exact. We then study the Koszul complex of an object V of the Verlinde category Ver_p ([19], Definition 3.1) and show that the only V for which $K^\bullet(V)$ is exact are classical supervector spaces. In other words, if $V = L_m$, where L_m is a non-invertible simple object of Ver_p (i.e., $2 \le m \le p-2$), then the symmetric algebra[1] SL_m is not Koszul; this happens for each $p \ge 5$.

We also generalize to the categorical setting the theory of Koszul algebras, and in particular Drinfeld's "Koszul deformation principle" stating that a homogeneous deformation of a Koszul algebra is flat if it is flat in degrees 2 and 3 [20]. Also, we show that if V is a Koszul object then the algebras SV and $\wedge V^*$ are Koszul. Finally, we show that if $2 \le m \le p-2$ then the algebras SL_m and $\wedge L_m$ are almost Koszul in the sense of [2], and for any $V \in \mathrm{Ver}_p$, compute the internal Ext-algebra of the augmentation module over SV and $\wedge V^*$ to itself, using the periodic Koszul complex of L_m (which, unlike the usual Koszul complex, is exact).

We then proceed to apply these results to the Poincare–Birkhoff–Witt theorem for Lie algebras in symmetric tensor categories. Again, in characteristic zero the generalization of the PBW theorem is straightforward, and several proofs extend easily to the categorical setting (e.g. the one based on the Campbell–Baker–Hausdorff formula, see [6], Exercise 9.9.7(viii), or the one based on the Koszul deformation principle, given below). On the other hand, in characteristic p it is not even clear what the right definition

[1] Throughout the paper, this algebra should not be confused with the special linear group, which we will denote by $\mathbf{SL}(m)$.

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