ARTICLE IN PRESS YAIMA:5898

Advances in Mathematics ••• (••••) •••–•••

Contents lists available at ScienceDirect

Advances in Mathematics

www.elsevier.com/locate/aim

# Special representations of Weyl groups: A positivity property

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ARTICLE INFO

*Article history:*
Available online xxxx

Dedicated to David Kazhdan on the occasion of his 70th birthday

*Keywords:*
Weyl group
Perron line
Special representation

ABSTRACT

The special representations of a Weyl group $W$ were introduced by the author in 1978. In this paper we show that the special representations of $W$, when viewed as representations of the corresponding J-ring, are characterized by a positivity property.



## 0. Introduction

Let $W$ be an irreducible Weyl group with length function $l : W \longrightarrow \mathbf{N}$ and let $S = \{s \in W; l(s) = 1\}$. Let Irr$W$ be a set of representatives for the isomorphism classes of irreducible representations of $W$ (over $\mathbf{C}$). In [7] a certain subset of Irr$W$ was defined. The representations in this subset were later called *special representations*; they play a key role in the classification of unipotent representations of a reductive group over a finite field $\mathbf{F}_q$ for which $W$ is the Weyl group. (The definition of special representations is reviewed in 3.1.)

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[1] Supported in part by National Science Foundation grant 1303060.

http://dx.doi.org/10.1016/j.aim.2017.06.015

It will be convenient to replace irreducible representations of $W$ with the corresponding simple modules of the asymptotic Hecke algebra $\mathbf{J}$ (see [11, 18.3]) associated to $W$ via the canonical isomorphism $\psi : \mathbf{C}[W] \xrightarrow{\sim} \mathbf{J}$ (see 3.1); let $E_\infty$ be the simple $\mathbf{J}$-module corresponding to $E \in \mathrm{Irr}W$ under $\psi$.

In this paper we show that a special representation $E$ of $W$ is characterized by the following positivity property of $E_\infty$: there exists a $\mathbf{C}$-basis of $E_\infty$ such that any element $t_u$ in the standard basis of $\mathbf{J}$ acts in this basis through a matrix with all entries in $\mathbf{R}_{\geq 0}$.

The fact that for a special representation $E$, $E_\infty$ has the positivity property above was pointed out (in the case where $W$ is of classical type) in [14]. In this paper I will recall the argument of [14] (see 3.3) and I give two other proofs which apply for any $W$. One of these proofs (see 4.4) is based on the interpretation [9,1], of $\mathbf{J}$ (or, rather, its part attached to a fixed two-sided cell) in terms of $G$-equivariant vector bundles on $X \times X$ where $X$ is a finite set with an action of a finite group $G$. Another proof (see Section 2) is based on the use of Perron's theorem for matrices with all entries in $\mathbf{R}_{>0}$. (Previously, Perron's theorem has been in used in the context of canonical bases in quantum groups in the study [10] of total positivity and, very recently, in the context of the canonical basis [4] of $\mathbf{C}[W]$, in [6]; in both cases the positivity properties of the appropriate canonical bases were used.) We also show that the Hecke algebra representation corresponding to a special representation $E$ can be realized essentially by a $W$-graph (in the sense of [4]) in which all labels are natural numbers. Some of our results admit also an extension to the case of affine Weyl groups (see Section 5).

## 1. Statement of the main theorem

**1.1.** Let $v$ be an indeterminate and let $\mathcal{A} = \mathbf{Z}[v, v^{-1}]$. Let $\mathcal{H}$ be the Hecke algebra of $W$ that is, the associative $\mathcal{A}$-algebra with 1 with an $\mathcal{A}$-basis $\{T_w; w \in W\}$ (where $T_1 = 1$) and with multiplication such that $T_w T_{w'} = T_{ww'}$ if $l(ww') = l(w) + l(w')$ and $(T_s + 1)(T_s - v^2) = 0$ if $s \in S$. Let $\{c_w; w \in W\}$ be the $\mathcal{A}$-basis of $\mathcal{H}$ denoted by $\{C'_w; w \in W\}$ in [4] (with $q = v^2$); see also [11, 5.2]. For example, if $s \in S$, we have $c_s = v^{-1}T_s + v^{-1}$. The left cells and two-sided cells of $W$ are the equivalence classes for the relations $\sim_L$ and $\sim_{LR}$ on $W$ defined in [4], see also [11, 8.1]; we shall write $\sim$ instead of $\sim_L$. For $x, y$ in $W$ we have $c_x c_y = \sum_{z \in W} h_{x,y,z} c_z$ where $h_{x,y,z} \in \mathbf{N}[v, v^{-1}]$. As in [11, 13.6], for $z \in W$ we define $a(z) \in \mathbf{N}$ by $h_{x,y,z} \in v^{a(z)}\mathbf{Z}[v^{-1}]$ for all $x, y$ in $W$ and $h_{x,y,z} \notin v^{a(z)-1}\mathbf{Z}[v^{-1}]$ for some $x, y$ in $W$. (For example, $a(1) = 0$ and $a(s) = 1$ if $s \in S$.) For $x, y, z$ in $W$ we have $h_{x,y,z} = \gamma_{x,y,z^{-1}} v^{a(z)} \mod v^{a(z)-1}\mathbf{Z}[v^{-1}]$ where $\gamma_{x,y,z^{-1}} \in \mathbf{N}$ is well defined. Let $\mathbf{J}$ be the $\mathbf{C}$-vector space with basis $\{t_w; w \in W\}$. For $x, y$ in $W$ we set $t_x t_y = \sum_{z \in W} \gamma_{x,y,z^{-1}} t_z \in \mathbf{J}$. This defines a structure of associative $\mathbf{C}$-algebra on $\mathbf{J}$ with unit element of the form $\sum_{d \in \mathcal{D}} t_d$ where $\mathcal{D}$ is a certain subset of the set of involutions in $W$, see [11, 18.3]. For any subset $X$ of $W$ let $\mathbf{J}_X$ be the subspace of $\mathbf{J}$ with basis $\{t_w; w \in X\}$; let $\mathbf{J}_X^+$ be the set of elements of the form $\sum_{w \in X} f_w t_w \in \mathbf{J}_X$ with $f_w \in \mathbf{R}_{>0}$ for all $w \in X$. We have $\mathbf{J} = \oplus_{\mathbf{c}} \mathbf{J}_{\mathbf{c}}$ where $\mathbf{c}$ runs over the two-sided cells of $W$. Each $\mathbf{J}_{\mathbf{c}}$