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New bounds in reduction theory of indefinite ternary integral quadratic forms

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Dedicated to D. Kazhdan on the occasion of his 70th birthday

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ABSTRACT

Using dynamics on homogeneous spaces we obtain some new and improved estimates for reduction of indefinite ternary integral quadratic forms.

1. Introduction and statement of results

1.1. Historical perspective

An important problem in reduction theory of integral quadratic forms is to decide to what extent one can simplify a given integral quadratic form by taking an equivalent

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form. In this context an explicit question is *given an integral quadratic form, how small can the coefficients of an equivalent form be?* Let Q be a quadratic form and $A = (a_{ij})$ the symmetric matrix of Q. Sometimes we will use $Q = (a_{ij})$ with abuse of notation. Recall that the quadratic form Q is called integral if the entries A are integers. To measure the complexity of Q we will consider $\det A$ the *determinant* of Q and

$$\mathrm{ht}(Q) := \max(|a_{ij}|)$$

the *height* of Q. Two integral quadratic forms are said to be equivalent if their symmetric matrices A and B satisfy $A = \gamma^t B\gamma$ for some $\gamma \in \mathrm{GL}_n(\mathbb{Z})$.

An early achievement in reduction theory, due to Lagrange, is that any non-degenerate binary integral quadratic form Q is equivalent to a form $ax^2+bxy+cy^2$ with $|b| \le |a| \le |c|$. This implies

$$\mathrm{ht}(ax^2 + bxy + cy^2) = |c| \le \frac{4}{3} \cdot |ac - \frac{b^2}{4}| = 4|\det Q|/3. \tag{1}$$

In other words, any binary quadratic form is equivalent to a form whose coefficients are all small. Since *equivalent integral quadratic forms have the same determinant* we have, by the finiteness of the choice of a, b and c, for any integer $D > 0$ there are only finitely many equivalence classes of binary quadratic forms with the determinant equal to D. Historically, an important application of reduction theory was to prove the finiteness for the number of equivalence classes of integral quadratic forms of a given determinant, in any number of variables.

1.2. Statement of results on quadratic forms

The goal of the present paper is to prove some new and improved estimates for reduction of indefinite ternary integral quadratic forms, using an approach based on the theory of homogeneous flows. For the dynamical setting we let, throughout the paper, G stand for $\mathrm{SL}_3(\mathbb{R})$ and Γ for $\mathrm{SL}_3(\mathbb{Z})$. We will fix the indefinite ternary integral quadratic form

$$Q_0(\boldsymbol{w}) = 2w_1w_3 - w_2^2 \qquad (\boldsymbol{w} \in \mathbb{R}^3),$$

and denote by $H = \mathrm{SO}(Q_0)^\circ_{\mathbb{R}}$ the identity component of the real points of the special orthogonal group of Q_0. We shall also fix a right invariant Riemannian metric on G which induces a G-invariant probability measure on G/Γ. The metric restricts to the closed subgroup H of G and gives rise a Haar measure m_H on H. For any indefinite ternary real quadratic form Q, there exists an element $g \in G$ such that

$$Q(\cdot) = (\det Q)^{\frac{1}{3}} \cdot Q_0(g\cdot). \tag{2}$$

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