

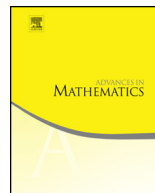


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# On the pro-semisimple completion of the fundamental group of a smooth variety over a finite field

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and admiration*MSC:*primary 14G15, 11G35  
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## ABSTRACT

Let  $\Pi$  be the fundamental group of a smooth variety  $X$  over  $\mathbb{F}_p$ . Let  $\overline{\mathbb{Q}}$  be an algebraic closure of  $\mathbb{Q}$ . Given a non-Archimedean place  $\lambda$  of  $\overline{\mathbb{Q}}$  prime to  $p$ , consider the  $\lambda$ -adic pro-semisimple completion of  $\Pi$  as an object of the groupoid whose objects are pro-semisimple groups and whose morphisms are isomorphisms up to conjugation by elements of the neutral connected component. We prove that this object does not depend on  $\lambda$ . If  $\dim X = 1$  we also prove a crystalline generalization of this fact.

We deduce this from the Langlands conjecture for function fields (proved by L. Lafforgue) and its crystalline analog (proved by T. Abe) using a reconstruction theorem in the spirit of Kazhdan–Larsen–Varshavsky.

We also formulate two related [Conjectures E.8.1 and E.9.1](#). Each of them is a kind of “reciprocity law” involving a sum over all  $\ell$ -adic cohomology theories (including the crystalline theory for  $\ell = p$ ).

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