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Donaldson–Thomas invariants of 2-dimensional sheaves inside threefolds and modular forms

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A B S T R A C T

Motivated by the S-duality conjecture, we study the Donaldson–Thomas invariants of the 2-dimensional Gieseker stable sheaves on a threefold. These sheaves are supported on the fibers of a nonsingular threefold X fibered over a nonsingular curve. In the case where X is a $K3$ fibration, we express these invariants in terms of the Euler characteristic of the Hilbert scheme of points on the $K3$ fiber and the Noether–Lefschetz numbers of the fibration. We prove that a certain generating function of these invariants is a vector modular form of weight $-3/2$ as predicted in S-duality.

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1. Introduction

1.1. Overview

We study the invariants virtually counting the configurations of a number of points and a vector bundle supported on the members of a system of divisors inside a nonsingular threefold. One of our motivations is that these invariants have been studied by the physicists [6,8,9,28,31] as a set of supersymmetric BPS invariants associated to D4-D2-D0 systems. By S-duality conjecture, the generating series of these invariants are expected to be modular.

In this paper we interpret these invariants by means of the moduli spaces of Gieseker stable coherent sheaves with 2-dimensional support inside a nonsingular threefold X. Another motivation for considering these moduli spaces is to find a sheaf-theoretic analog of the formulas proven in [25] that relate the Gromov–Witten invariants of a threefold to the Gromov–Witten invariants of a system of its divisors.

1.2. Moduli spaces of 2-dimensional sheaves

Let X be a nonsingular projective threefold over $\mathbb{C}$ with a fixed polarization L. For a given nonzero effective irreducible divisor class $F \in \operatorname{Pic}(X)$, we fix a Chern character vector

$$\mathsf{ch} = (0, rF, \gamma, \mathsf{ch}_3) \in \oplus_{i=0}^{3} H^{2i}(X, \mathbb{Q}), \tag{1}$$

with $r > 0$. If $\mathcal{F}$ is a coherent sheaf with $\operatorname{ch}(\mathcal{F}) = \mathsf{ch}$ then, $\mathcal{F}$ is supported on some divisor with the numerical class $r'F$ where $r' \mid r$. We always assume that any Gieseker L-semistable sheaf $\mathcal{F}$ with $\operatorname{ch}(\mathcal{F}) = \mathsf{ch}$ is stable.

We consider the moduli space $\mathcal{M} = \mathcal{M}^L(X, \mathsf{ch})$ of Gieseker L-semistable sheaves with Chern character ch. By our assumption, $\mathcal{M}$ is projective and the geometric points correspond to the isomorphism classes of stable (hence pure) 2-dimensional sheaves with Chern character ch. It is proven in [30] that $\mathcal{M}$ admits a perfect obstruction theory if

$$\operatorname{Ext}^3(\mathcal{F}, \mathcal{F})_0 = 0$$

for any geometric point $\mathcal{F} \in \mathcal{M}$, where the index 0 indicates the trace free part of $\operatorname{Ext}^3(\mathcal{F}, \mathcal{F})$. In this case, Thomas constructs (Theorem 2.3) a natural perfect obstruction theory $E^\bullet \to L^\bullet_{\mathcal{M}}$ such that

$$h^i(E^{\bullet\vee})_{\mathcal{F}} \cong \operatorname{Ext}^{i+1}(\mathcal{F}, \mathcal{F})$$

for $i = 0, 1$. By [3,21] one obtains a virtual cycle

$$[\mathcal{M}]^{vir} = [\mathcal{M}, E^\bullet]^{vir} \in A_{\mathrm{vd}}(\mathcal{M})$$