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On stability of Abrikosov vortex lattices

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A B S T R A C T

The Ginzburg–Landau equations play a key role in superconductivity and particle physics. They inspired many imitations in other areas of physics. These equations have two remarkable classes of solutions – vortices and (Abrikosov) vortex lattices. For the standard cylindrical geometry, the existence theory for these solutions, as well as the stability theory of vortices are well developed. The latter is done within the context of the time-dependent Ginzburg–Landau equations – the Gorkov–Eliashberg–Schmid equations of superconductivity – and the abelian Higgs model of particle physics.

We study stability of Abrikosov vortex lattices under finite energy perturbations satisfying a natural parity condition (both defined precisely in the text) for the dynamics given by the Gorkov–Eliashberg–Schmid equations. For magnetic fields close to the second critical magnetic field and for arbitrary lattice shapes, we prove that there exist two functions on the space of lattices, such that Abrikosov vortex lattice solutions are asymptotically stable, provided the superconductor is of Type II and these functions are positive, and unstable, for superconductors of Type I, or if one of these functions is negative.

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1. Introduction

1.1. Problem and results

The macroscopic theory of superconductivity is a crown achievement of condensed matter physics, presented in any book on the subject. It was developed along the lines of Landau's theory of the second order phase transitions before the microscopic theory was discovered. At the foundation of this theory lie the celebrated Ginzburg–Landau equations,

$$\begin{cases} -\Delta_A \Psi - \kappa^2(1-|\Psi|^2)\Psi = 0, \\ \operatorname{curl}^* \operatorname{curl} A - \operatorname{Im}(\bar{\Psi}\nabla_A \Psi) = 0, \end{cases} \tag{1.1}$$

which describe superconductors in thermodynamic equilibrium. Here Ψ is a complex-valued function, called the order parameter, A is a vector field (the magnetic potential), κ is a positive constant, called the Ginzburg–Landau parameter, $\nabla_A = \nabla - iA$ and $\Delta_A = \nabla_A \cdot \nabla_A$ are the covariant gradient and Laplacian. Physically, $|\Psi|^2$ gives the (local) density of superconducting electrons (Cooper pairs), $B = \operatorname{curl} A$ is the magnetic field. The second equation is Ampère's law with $J_S = \operatorname{Im}(\bar{\Psi}\nabla_A \Psi)$ being the supercurrent associated to the electrons having formed Cooper pairs.

We assume, as common, that superconductors fill in all of $\mathbb{R}^2$ (the cylindrical geometry in $\mathbb{R}^3$). In this case,

$$\operatorname{curl} A := \partial_{x_1} A_2 - \partial_{x_2} A_1 \quad \text{and} \quad \operatorname{curl}^* f = (\partial_{x_2} f, -\partial_{x_1} f).$$

By far, the most important and celebrated solutions of the Ginzburg–Landau equations are magnetic vortex lattice solutions, discovered by Abrikosov ([1]), and known as (Abrikosov) vortex lattice solutions or simply *Abrikosov or vortex lattices*. They describe the free energy in the presence of magnetic fields, and understanding these solutions is important for maintaining the superconducting current in Type II superconductors, i.e., for $\kappa > \frac{1}{\sqrt{2}}$.

Abrikosov lattices have been extensively studied in the physics literature. Among many rigorous results, we mention that the existence of these solutions was proven rigorously in [31,10,15,7,44,40]. Moreover, important and fairly detailed results on asymptotic behaviour of solutions, for $\kappa \to \infty$ and applied magnetic fields, h, satisfying $h \leq \frac{1}{2}\ln\kappa + \text{const}$ (the London limit), were obtained in [8] (see this paper and the book [36] for references to earlier work). Further extensions to the Ginzburg–Landau equations for anisotropic and high temperature superconductors in the $\kappa \to \infty$ regime can be found in [4–6]. (See [14,26,38] for reviews.)

In this paper we are interested in the dynamics of the Abrikosov lattices, as described by the time-dependent generalization of the Ginzburg–Landau equations proposed by Schmid ([37]) and Gorkov and Eliashberg ([19]) (earlier versions are due to Bardeen