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A remark on Gromov–Witten invariants of quintic threefold

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A R T I C L E   I N F O

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A B S T R A C T

The purpose of the article is to give a proof of a conjecture of Maulik and Pandharipande for genus 2 and 3. As a result, it gives a way to determine Gromov–Witten invariants of the quintic threefold for genus 2 and 3.

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1. Introduction

Let Q be the quintic threefold in $\mathbb{P}^4$. $\mathbb{P}(N_{Q/\mathbb{P}^4} \oplus \mathcal{O}_Q)$ is the projective bundle associated to the vector bundle $N_{Q/\mathbb{P}^4} \oplus \mathcal{O}_Q$ over Q. D_0 is a divisor of $\mathbb{P}(N_{Q/\mathbb{P}^4} \oplus \mathcal{O}_Q)$ determined by the factor $N_{Q/\mathbb{P}^4}$.

Gathmann [12] used relative virtual localization technique to reduce some relative Gromov–Witten invariants of the pair $(\mathbb{P}(N_{Q/\mathbb{P}^4} \oplus \mathcal{O}_Q), D_0)$ to the absolute Gromov–Witten invariants of Q when genus $g \leq 1$. Combining it with degeneration formula (2.7), which relates Gromov–Witten invariants of $\mathbb{P}^4$ to relative invariants of the pairs $(\mathbb{P}^4, Q)$ and $(\mathbb{P}(N_{Q/\mathbb{P}^4} \oplus \mathcal{O}_Q), D_0)$, he could recursively determine Gromov–Witten invariants of the quintic threefold $N_{g,d}$ (3.2) for genus $g \leq 1$. For a discussion of the history of computing Gromov–Witten invariants of quintic threefold, we recommend the reader to see [24], [26].

Later, Maulik and Pandharipande have found an algorithm (see [27], Theorem 1) to determine relative invariants of the pair $(\mathbb{P}(N_{Q/\mathbb{P}^4} \oplus \mathcal{O}_Q), D_0)$ from the absolute invariants of Q without the constraint of genus. Inspired by Gathmann's proposal, they proposed the following conjecture:

Conjecture 1.1 *([27]). The system of equations obtained from the degeneration formula (2.7) (set $(V, W) = (\mathbb{P}^4, Q)$ in the formula) and the Maulik–Pandharipande's algorithm (see Section 2.4 or [27], Theorem 1) can be used to determine both the relative theory of the pair $(\mathbb{P}^4, Q)$ and the Gromov–Witten invariants $N_{g,d}$ of Q.*

Remark 1.2. Conjecture 1.1 for $g = 0, 1$ directly follows from the idea of Gathmann. Maulik and Pandharipande have claimed in their paper that they have proven Conjecture 1.1 for genus 2, but they did not give a proof.

In this paper, we prove that

Theorem 1.3. *The Conjecture 1.1 is true for $g = 2, 3$.*

As a consequence of Theorem 1.3, it gives an algorithm to determine $N_{g,d}$ for $g = 2, 3$. Here, we do not claim any priority to the proof of Conjecture 1.1 for genus 2. We may owe it to Maulik and Pandharipande.

Remark 1.4. In the same paper [27], Maulik and Pandharipande also gave a calculation scheme to determine all $N_{g,d}$, which is different from the method of Conjecture 1.1. But